Second Order Elliptic Operators on Triple Junction Surfaces

Gaoming Wang

Cornell University

Oct 03 2022

Contents

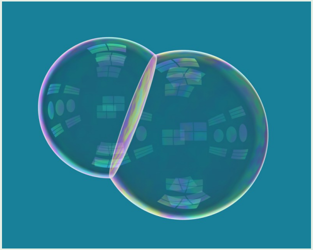
Outline

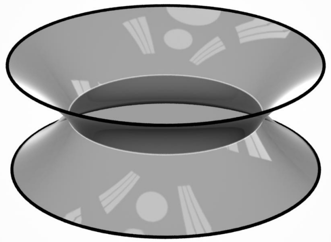
Definition (Triple Junction Surfaces)

Let $(\Sigma^i, \partial\Sigma^i)_{i=1,2,3}$ be three surfaces with boundary in an ambient manifold M. We call the union $\Sigma = \cup_{i=1}^3 \Sigma^i$ is a *triple junction surface* if these surfaces Σ^i have the same boundary, i.e., $\partial\Sigma^1 = \partial\Sigma^2 = \partial\Sigma^3$. We write Γ as their common boundary $\Gamma = \partial\Sigma^i$ for $i = 1,2,3$.

We will suppose Γ is connected for simplicity.

Examples (https://faculty.math.illinois.edu/~jms/Images/)





Definition (Minimal triple junction surfaces)

A triple junction surface Σ is called a *minimal triple junction surface* if the follows hold.

- $H_{\Sigma^i} = 0$ for each $1 \le i \le 3$.
- $\angle(\tau^i, \tau^j) = \frac{2\pi}{3}$, for $1 \le i \ne j \le 3$.

where the τ^i is the outer conormal of Σ^i.
Since Γ is connected, we can choose $\nu = (\nu^1, \nu^2, \nu^3)$ such that $\sum_{i=1}^3 \nu^i = 0$ along Γ.

Examples

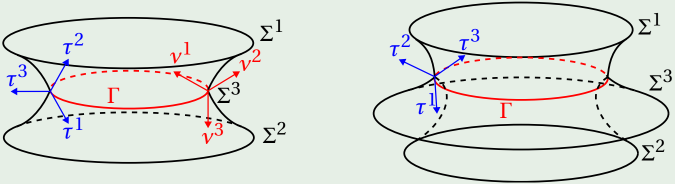


Figure: Two kinds of Y-shaped catenoid in $\mathbb{R}^3$

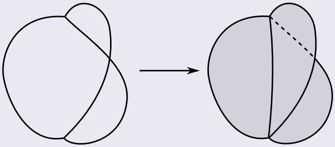
Some Results Related to Minimal Triple Junction Surfaces

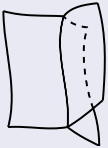
Existence Problem

- **C. Mese, S. Yamada (2006):** Solving Plateau problem with Y-type singularities.
  Giving suitable networks, find the minimal triple junction disk by minimizing Dirichlet energy.

Regularity

- **L. Simon (1993):** $C^{1,\alpha}$ regularity.
  If a stationary integral varifold V is sufficient close to a union of three half planes near p, then V has $C^{1,\alpha}$ regularity.
- **B. Krummel (2014):** Higher regularity of minimal submanifolds with common boundary.





Stationary varifold

↓

$C^{1,\alpha}$ regularity

↓

Smooth (Analytic)

Rigidity

- **J. Bernstein, F. Maggi (2021):** The rigidity of the Y-shaped catenoid.
  If a minimal triple junction surface Σ has two catenoid-like ends, then it is isometric to a Y-catenoid. (Method, Alexandrov's method of moving planes.)

Equivalent Definitions of Minimal Triple Junction Surfaces in $\mathbb{R}^3$

Proposition (Equivalent definitions)

A triple junction surface Σ is a minimal triple junction surface in $\mathbb{R}^3$ if and only if one of the following holds.

- Σ is a critical point of area function. (This holds for the general ambient manifold.)
- The coordinate the function x_j^i for $j = 1, 2, 3$ solve the following PDE problem.

$$\begin{cases} \Delta_{\Sigma^i} x_j^i = 0, & \text{in } \Sigma^i, \\ x_j^1 = x_j^2 = x_j^3, & \text{on } \Gamma, \\ \sum_{i=1}^3 \frac{\partial x_j^i}{\partial \tau^i} = 0, & \text{on } \Gamma. \end{cases}$$

First variation formula

$$\delta |\Sigma| (X) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Area}(\Sigma_t) = \sum_{i=1}^3 \int_{\Sigma^i} \mathrm{div}_{\Sigma^i} X dA = -\sum_{i=1}^3 \int_{\Sigma^i} \vec{H}_{\Sigma^i} \cdot X dA + \sum_{i=1}^3 \int_\Gamma X \cdot \tau^i ds$$

Outline

Notations and assumption

- From now on, for the triple junction surface Σ, we will only consider each Σ^i is orientable and has only one boundary component $\partial\Sigma^i = \Gamma$.
- We will choose ν^i be the normal vector field on Σ^i such that $\sum_{i=1}^3 \nu^i = 0$ along Γ. Write $\nu = (\nu^1, \nu^2, \nu^3)$ for short.

Integral convention

- For any $f = (f^1, f^2, f^3)$, we use the following short notations.

$$\int_\Sigma f := \sum_{i=1}^{3} \int_{\Sigma^i} f^i \, dA^i \qquad \int_\Gamma f := \int_\Gamma \sum_{i=1}^{3} f^i \, ds_\Gamma$$

- Example. We can write the first variation formula as

$$\delta |\Sigma| (X) = -\int_\Sigma H_\Sigma \phi + \int_\Gamma X \cdot \tau$$

for $\phi = (\phi^1, \phi^2, \phi^3) = (\langle X, \nu^1 \rangle, \langle X, \nu^2 \rangle, \langle X, \nu^3 \rangle) := \langle X, \nu \rangle$.

Second Variation Formula and Stability

Second variation formula for minimal triple junction surfaces

Let Σ be a minimal triple junction surface in M, then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \mathrm{Area}(\Sigma_t) = \int_\Sigma \left[\left|\nabla_\Sigma \phi\right|^2 - |A_\Sigma|^2 \phi^2 - \mathrm{Ric}^M(\nu)\phi^2 \right] d\Sigma - \boxed{\int_\Gamma \phi^2 \langle \boldsymbol{H}_\Gamma, \tau \rangle \, d\Gamma}$$

with $\phi = \langle X, \nu \rangle$ which has compact support, $\boldsymbol{H}_\Gamma$, the geodesic curvature vector of Γ in M.

Stable minimal triple junction surfaces

We say $\Sigma \subset M$ is a stable minimal triple junction surface if $\frac{\mathrm{d}^2}{\mathrm{d}t^2}|_{t=0}\mathrm{Area}(\Sigma_t) \geq 0$ for any variation Σ_t of Σ with compact support.

Stability inequality

If Σ is stable, then for any $\phi \in W^{1,2}(\Sigma^1) \times W^{1,2}(\Sigma^2) \times W^{1,2}(\Sigma^3)$ with compact support,

$$\int_\Sigma (|A_\Sigma|^2 + \mathrm{Ric}^M(\nu))\phi^2 + \int_\Gamma \phi^2 \langle \boldsymbol{H}_\Gamma, \tau \rangle \leq \int_\Sigma \left|\nabla_\Sigma \phi\right|^2 \text{ for all } \boxed{\sum_{i=1}^{3} \phi^i = 0} \text{ on } \Gamma \text{ in the trace sense.}$$

Theorem

Let Σ be a minimal triple junction surface in $\mathbb{R}^3$. Suppose Σ is complete, stable and has quadratic area growth. Then Σ^i is flat.

Remark

Since we have assumed each Σ^i is connected and Γ is connected, the only possible Σ satisfying above theorem is the union of three half planes intersecting each other at $\frac{2\pi}{3}$ along their common boundaries.

Sketch Proof of Generalized Bernstein Theorem

Proof. Using Logarithmic cutoff functions.

- Fix three nonzero distinct constants c^i such that $\sum_{i=1}^3 c^i = 0$ and fix a point $p \in \Gamma$. $\rho(q) := |q - p|$ denotes the distance from p for $q \in \Sigma$. Choose ϕ as

$$\phi^i = \begin{cases} c^i, & q \in B_1(p) \cap \Sigma^i \\ c^i \left(1 - \frac{\log \rho}{n}\right), & q \in (B_{e^n}(p) \backslash B_1(p)) \cap \Sigma^i, \\ 0, & \text{otherwise.} \end{cases}$$

- Stability inequality implies

$$\int_{B_1(p)\cap\Sigma} c^2 |A_\Sigma|^2 + \boxed{\int_\Gamma \phi^2 \langle \boldsymbol{H}_\Gamma, \nu \rangle} \le \int_{(B_{e^n}(p)\backslash B_1(p))\cap\Sigma} \frac{c^2}{n^2 \rho^2} \le \frac{C}{n} \sum_{i=1}^3 (c^i)^2.$$

- By replacing $(c^1, c^2, c^3) \to \frac{1}{\sqrt{3}}(c^2 - c^3, c^3 - c^1, c^1 - c^2)$, we can change the sign of $\int_\Sigma \phi^2 \langle \boldsymbol{H}_\Gamma, \nu \rangle$. So, we can choose suitable c s.t. the highlight term is non-negative.
- Let $n \to +\infty$, we get A_Σ vanishes in $B_1(p) \cap \Sigma$. So A_Σ vanishes everywhere.

□

Outline

Useful Function Spaces

Subbundles of $\Gamma \times \mathbb{R}^3$

- We write $E \subset \Gamma \times \mathbb{R}^3$ as the smooth rank k subbundle of $\Gamma \times \mathbb{R}^3$.
- Two important subbundles

$$E_1 := \left\{(p,g) \in \Gamma \times \mathbb{R}^3 : g^1 = g^2 = g^3\right\},$$
$$E_2 := \left\{(p,g) \in \Gamma \times \mathbb{R}^3 : g^1 + g^2 + g^3 = 0\right\}.$$

- Complemented subbundles. Given rank k subbundle E of $\Gamma \times \mathbb{R}^3$, we write

$$E^\perp := \left\{(p,g) \in \Gamma \times \mathbb{R}^3 : g \cdot g_1 = 0 \text{ for any } (p,g_1) \in E\right\}$$

where the dot means the inner product in $\mathbb{R}^3$. In particular, $E_1^\perp = E_2$.

Function spaces

$$C_E^k(\Sigma) := \{f = (f^1, f^2, f^3) : f^i \in C^k(\Sigma^i) \text{ and } f|_\Gamma \text{ is a section of } E\}.$$
$$W_E^{k,p}(\Sigma) := \{f = (f^1, f^2, f^3) : f^i \in W^{k,p}(\Sigma^i) \text{ and } f|_\Gamma \text{ is a section of } E,\ \mu_\Gamma\text{-a.e.}\}.$$

Useful Function Spaces

$C^k_{E_1}(\Sigma)$ and $W^{k,p}_{E_1}(\Sigma)$ (Restrict function to Σ)

If $f_0 \in C^k(M)$, then $f := f_0|_\Sigma \in C^k_{E_1}(\Sigma)$ since $f^1 = f^2 = f^3$ along Γ.
In particular, coordinate function $x_j = (x_j^1, x_j^2, x_j^3) \in C^\infty_{E_1}(\Sigma)$ if Σ is a triple junction surface in $\mathbb{R}^3$.

$C^k_{E_2}(\Sigma)$ and $W^{k,p}_{E_2}(\Sigma)$ (Inner product of vector fields.)

If $V \in \mathfrak{X}^k(M)$, then $f := \langle V, \nu \rangle \in C^k_{E_2}(\Sigma)$ since $\sum_{i=1}^3 f^i = \langle V, \sum_{i=1}^3 \nu^i \rangle = 0$ along Γ.
In particular, we know the stability inequality holds for $\phi \in W^{1,2}_{E_2}(\Sigma)$ if Σ is stable.

Remark

If Γ has more than one component, in general, we cannot choose ν on Σ such that $\sum_{i=1}^3 \nu^i = 0$ along Γ. In this case, we need to choose another E instead of E_2 to make sure $\phi \in W^{1,2}_E(\Sigma)$.

Elliptic PDE problem (Suppose Σ is compact)

Define $L = \mathrm{div}_\Sigma(J \cdot \nabla^\Sigma u) + cu$, $h = (h^1, h^2, h^3) \in C^\infty(\Gamma, \mathbb{R}^3)$. Here, J is a symmetric elliptic 2-tensor on Σ and $c = (c^1, c^2, c^3)$ such that c^i is a smooth function on $\overline{\Sigma^i}$. Given f, we consider the problem,

$$\begin{cases} -Lu = f, & \text{in } \Sigma, \\ u|_\Gamma \text{ is a section of } E, & \\ J(\nabla^\Sigma u, \tau) + uh \text{ is a section of } E^\perp. & \end{cases} \qquad (\star)$$

We denote (L, h, E) the second order elliptic operator on Σ.

Bilinear form and weak solution

We write the bilinear form $B[\cdot,\cdot]$ associated with operator (L, h, E) by

$$B[u,v] = \int_\Sigma \left[J(\nabla^\Sigma u, \nabla^\Sigma v) - cuv \right] + \int_\Gamma uvh.$$

We say $u \in W^{1,2}_E(\Sigma)$ is a *weak solution* to $(\star)$ if $B[u,v] = \int_\Sigma fv$ for all $v \in W^{1,2}_E(\Sigma)$.

Some Examples of Important Operators

Minimal triple junction surfaces in $\mathbb{R}^3$

A triple junction surface Σ is minimal $\in \mathbb{R}^3$ if and only if each coordinate function x_j solves Problem $(\star)$ with elliptic operator $(\Delta_\Sigma, 0, E_1)$ and $f = 0$ for $j = 1, 2, 3$.

Minimal triple junction surfaces in $\mathbb{S}^3$

A triple junction surface Σ in $\mathbb{S}^3$ is minimal if and only if each coordinate function x_j is an eigenfunction of operator $(\Delta_\Sigma, 0, E_1)$ with eigenvalue 2 for $j = 1, 2, 3, 4$.
In other words, x_j solves Problem $(\star)$ with operator $(\Delta_\Sigma, 0, E_1)$ and $f = 2x_j$.

Stability operators

From second variational formula, we define stability form $S(\cdot,\cdot)$ by

$$S(u,v) := \int_\Sigma \langle \nabla^\Sigma u, \nabla^\Sigma v\rangle - [|A_\Sigma|^2 + \mathrm{Ric}^M(\nu)]\, uv - \int_\Gamma \langle \boldsymbol{H}_\Gamma, \tau\rangle\, uv, \forall u, v \in W^{1,2}_{E_2}(\Sigma).$$

Stability form $S(\cdot,\cdot)$ is a bilinear form with operator $(\Delta_\Sigma + |A_\Sigma|^2 + \mathrm{Ric}^M(\nu), -\boldsymbol{H}_\Gamma \cdot \tau, E_2)$.

Existence and Regularity of Weak Solutions

Theorem (Existence (Special case))

If Problem $(\star)$ only has a trivial solution when $f \equiv 0$, then it has a unique weak solution u for any $f \in L^2(\Sigma)$.

Theorem (Regularity)

Suppose u is a weak solution to Problem $(\star)$ and $E = E_1$ or E_2. If $f \in W^{m,2}(\Sigma)$, then $u \in W^{m+2,2}(\Sigma)$.
In particular, if $f \in C^\infty(\Sigma)$, then $u \in C_E^\infty(\Sigma)$ and u is a classical solution to Problem $(\star)$.

Proof.

- Existence. An easy application of Fredholm alternative.
- Regularity. Quite similar to the regularity of PDE problems on boundary.

□

Outline

Eigenvalues and Eigenfunctions of Elliptic Operator (L, h, E)

Definition

We say a function $u \in C_E^\infty(\Sigma)$ is an eigenfunction corresponding to the eigenvalue λ to the operator (L, h, E) if u solves Problem $(\star)$ with operator (L, h, E) and $f = \lambda u$.

Proposition

If we arrange the eigenvalues of (L, h, E) as $\lambda_1 \leq \lambda_2 \leq \cdots$, then we can use a variational formula to characterize eigenvalues as

$$\lambda_k = \min_{V_k \subset W_E^{1,2}(\Sigma)} \max_{u \in V_k} \frac{B[u,u]}{\|u\|^2_{L^2(\Sigma)}},$$

where V_k is any k-dimensional subspace of $W_E^{1,2}(\Sigma)$.

Definition (The operator is given by $(\Delta_\Sigma + |A_\Sigma|^2 + \mathrm{Ric}^M(\nu), -\langle \boldsymbol{H}_\Gamma, \nu\rangle, E_2)$)

The (Morse) index and nullity of Σ is defined as,

$$\mathrm{Ind}(\Sigma) := \text{the largest } k \text{ s.t. } \lambda_k < 0. \quad \mathrm{Nul}(\Sigma) := \text{the number of } \lambda_k \text{ s.t. } \lambda_k = 0.$$

Dirichlet-to-Neumann maps (Special case, $\mathrm{Nul}(\Sigma^i) = 0$ for each $1 \le i \le 3$)

Fix a triple junction surface Σ and an operator (L, h, E). For any $g \in C^\infty(\Gamma)$, let $u_g^i \in C^\infty(\Sigma^i)$ with $L(u_g^i) = 0$ and $u_g^i|_\Gamma = g$. (u_g^i is called the L-extension of g on Σ^i.) We define the Dirichlet-to-Neumann map T^i on boundary of Σ^i as,

$$T^i : C^\infty(\Gamma) \to C^\infty(\Gamma), \quad T^i(g) = J(\nabla^{\Sigma^i} u^i, \tau^i) + h^i g.$$

Now we can define the Dirichlet-to-Neumann map $T : C_E^\infty(\Gamma) \to C_E^\infty(\Gamma)$ by

$$T(g^1, g^2, g^3) = P_E\left(\left(T^i(g^i)\right)_{i=1}^3\right),$$

where P_E is the orthogonal projection of vector bundle $\Gamma \times \mathbb{R}^3 \to E$.

Proposition

The Dirichlet-to-Neumann map T is a compact self-adjoint operator on $C_E^\infty(\Gamma)$. Hence, $\mathrm{Ind}(T), \mathrm{Nul}(T)$ are all well-defined.

Index and nullity theorem (Special case $\mathrm{Nul}(\Sigma^i) = 0$)

The Morse index for Σ can be computed by

$$\mathrm{Ind}(\Sigma) = \sum_{i=1}^{3} \mathrm{Ind}(\Sigma^i) + \mathrm{Ind}(T).$$

The nullity of Σ can be computed by $\mathrm{Nul}(\Sigma) = \mathrm{Nul}(T)$.

Proof.

We can use the Min-Max characterization of eigenvalues for these operators to finish the proof.
Indeed, the above theorem holds even when some of $\mathrm{Nul}(\Sigma^i) \neq 0$. □

Quick application

Let Σ be the standard minimal triple junction sphere in $\mathbb{S}^3$, then $\mathrm{Ind}(\Sigma) = 2$ and $\mathrm{Nul}(\Sigma) = 5$.

Index and Nullity for the Tetrahedron-type network

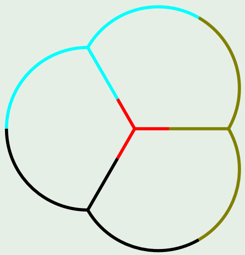
If $\mathcal{N}$ is the geodesic networks of tetrahedron-type, then

$$\mathrm{Ind}(\mathcal{N}) = 3, \quad \mathrm{Nul}(\mathcal{N}) = 3.$$

Ideas of computing index and nullity of $\mathcal{N}$.

We can find $\mathrm{Ind}(\mathcal{N}) \geq 3$ and $\mathrm{Nul}(\mathcal{N}) \geq 3$ easily.

Example



Using D-N maps

We cut $\mathcal{N}$ as showed in the left image. Each sub-network is stable $(\mathrm{Ind}(\mathcal{N}_i) = 0, \mathrm{Nul}(\mathcal{N}_i) = 0)$.
Let T be the Dirichlet-to-Neumann map on $\mathbb{R}^6$. Then

$$\mathrm{Ind}(\mathcal{N}) = \mathrm{Ind}(T), \quad \mathrm{Nul}(\mathcal{N}) = \mathrm{Nul}(T).$$

This will imply $\mathrm{Ind}(\mathcal{N}) + \mathrm{Nul}(\mathcal{N}) \leq 6$.

Thank You!