Minimal Hypersurfaces

Stability, Regularity, and Applications

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MEAN CURVATURE

$H = 0$

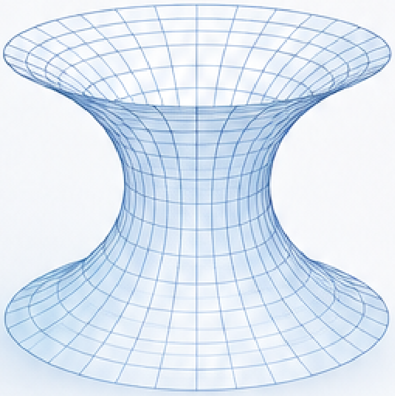
STABILITY

$Q(u,u) = \int_\Sigma \left(|\nabla u|^2 - (|A|^2 + Ric(\nu,\nu))u^2 \right) d\mu \geq 0$

JACOBI OPERATOR

$Lu = \Delta_\Sigma u + (|A|^2 + \mathrm{Ric}(\nu,\nu))u$

CATENOID

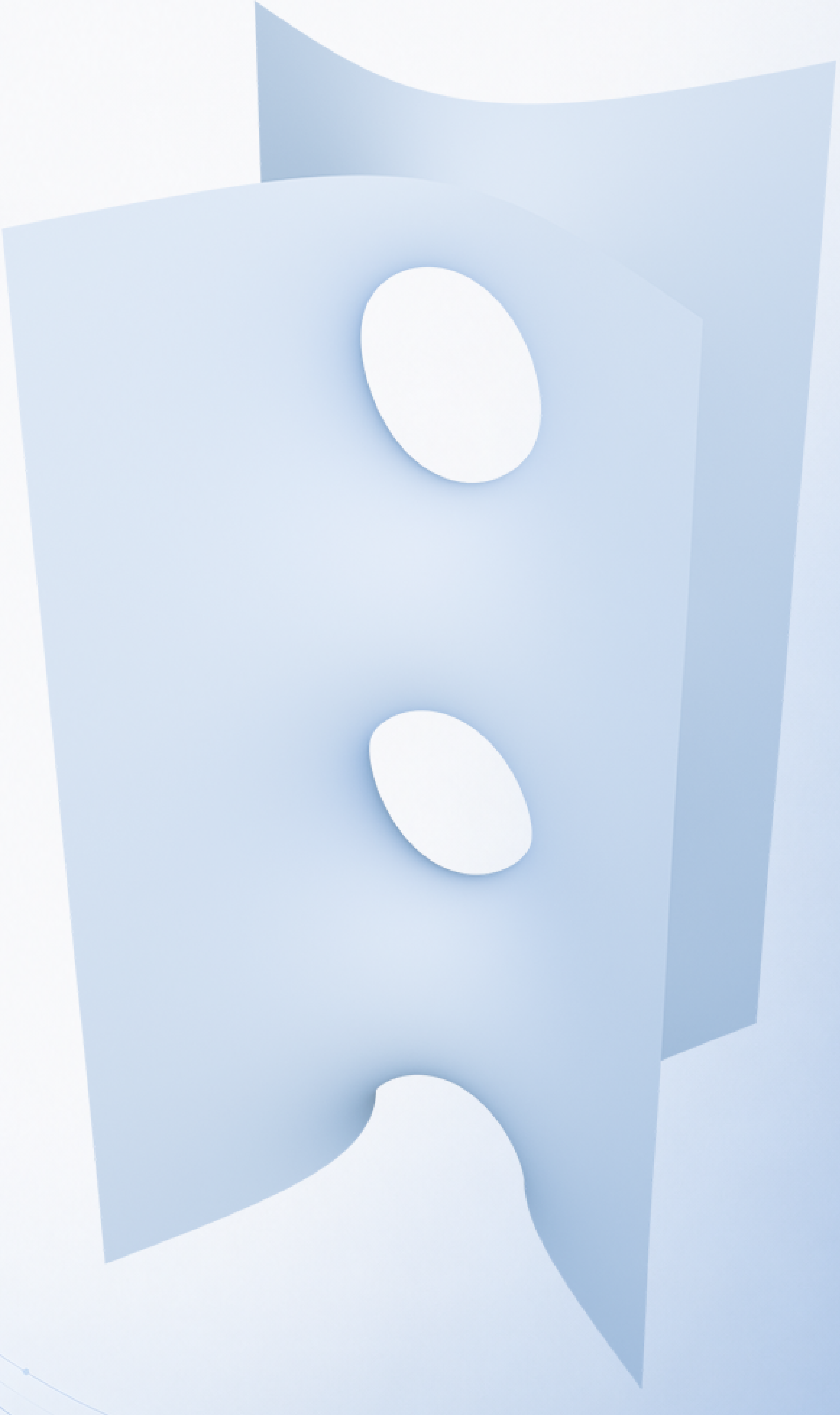


MINIMAL SURFACE EQUATION

$$\mathrm{div}_\Sigma \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0$$

MONOTONICITY FORMULA

$$\frac{d}{dr} \left(r^{-n}\, \mathrm{Area}(\Sigma \cap B_r(p)) \right) \geq 0$$



Preface

These notes are based on a course on minimal hypersurfaces given at the Beijing Institute of Mathematical Sciences and Applications (BIMSA). The goal is to give a concise introduction to several central ideas in the theory of minimal hypersurfaces, with an emphasis on stability, curvature estimates, regularity, and applications to scalar curvature and mathematical general relativity.

The material begins with basic Riemannian geometry and the first and second variation formulas. It then discusses geometric measure theory, regularity and compactness theorems for stable minimal hypersurfaces, stable Bernstein-type problems, and recent applications of minimal hypersurface methods to positive scalar curvature, the positive mass theorem, and related rigidity questions. The exposition is intended to be self-contained enough for graduate students in geometry, while still keeping the main analytic and geometric mechanisms visible.

Comments, corrections, and suggestions are welcome and may be sent to wanggaoming@bimsa.cn.

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Chapter 1 Background on Riemannian Manifolds and Minimal Immersions

1.1 Riemannian Manifolds

Definition 1.1.1 (Riemannian manifold)

*A **Riemannian manifold** is a smooth manifold M equipped with a Riemannian metric g, which is a smooth, positive-definite symmetric $(0,2)$-tensor field.* ♣

We use either $g(X,Y)$ or $\langle X,Y\rangle$ to denote the inner product of two tangent vectors $X, Y \in TM$ with respect to the metric g.

The Riemann tensor $R = R_M$ is defined by

$$R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z,$$

Here, ∇ is the Levi-Civita connection associated with g, i.e., it satisfies the following properties:

- *Torsion-free*: $\nabla_X Y - \nabla_Y X = [X,Y]$ for all vector fields X, Y.
- *Metric compatibility*: $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for all vector fields X, Y, Z.

The convention for the $(0,4)$-type Riemann curvature tensor is

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

If the ambient manifold is Euclidean space $\mathbb{R}^N$ with the standard metric, we write $X \cdot Y$ for the inner product of X and Y, and D for the standard connection.

1.2 Submanifolds

Definition 1.2.1 (Induced metric)

*Let (M^N, g) be a Riemannian manifold and $\iota : \Sigma^n \hookrightarrow M$ be a smooth immersion. The **induced metric** on Σ is defined by*

$$\bar{g} = \iota^* g,$$

so that $\bar{g}(X,Y) = g(d\iota(X), d\iota(Y))$ for tangent vectors $X, Y \in T\Sigma$. ♣

Definition 1.2.2 (Second fundamental form and mean curvature)

*The **vector-valued second fundamental form** $\vec{A}$ is defined by*

$$\vec{A}(X,Y) = (\nabla_X Y)^\perp,$$

*where ∇ is the Levi-Civita connection of M and $(\cdot)^\perp$ denotes the projection onto the normal bundle of Σ in M. The **mean curvature vector** $\vec{H}$ is defined as the trace of $\vec{A}$:*

$$\vec{H} = \mathrm{Tr}(\vec{A}) = \sum_{i=1}^{n} \vec{A}(e_i, e_i),$$

where $\{e_i\}$ is an orthonormal basis of $T\Sigma$ with respect to the induced metric $\bar{g}$. ♣

Definition 1.2.3 (Hypersurface and principal curvatures)

A submanifold Σ *is called a* ***hypersurface*** *when* $\operatorname{codim}(\Sigma) = 1$, *i.e.,* $n = N - 1$.

If we assume Σ *is a two-sided hypersurface, then there exists a globally defined unit normal vector field* ν *on* Σ*. In this case, the second fundamental form can be expressed as a scalar-valued symmetric* $(0, 2)$*-tensor* A *defined by*

$$A(X, Y) = g(\vec{A}(X, Y), \nu) = -g(\nabla_X \nu, Y) = g(\nabla_X Y, \nu).$$

Then the mean curvature $H = g(\vec{H}, \nu)$ *is the trace of* A.

The eigenvalues $\kappa_1, \ldots, \kappa_n$ *of* A *are called the* ***principal curvatures****.*

Gauss equation for hypersurfaces.

For a hypersurface Σ in a Riemannian manifold (M^N, g), the Gauss equation relates the intrinsic curvature of Σ to the extrinsic curvature and the ambient curvature:

$$R_\Sigma(X, Y, Z, W) = R_M(X, Y, Z, W) + A(X, Z)A(Y, W) - A(X, W)A(Y, Z),$$

1.3 First and Second Variation Formulas

We consider a variation of the submanifold Σ given by a family of immersions $\iota_t : \Sigma \to M$ with $\iota_0 = \iota$. The variation vector field is defined as

$$V = \left.\frac{\partial \iota_t}{\partial t}\right|_{t=0}.$$

The volume element of $\Sigma_t = \iota_t(\Sigma)$ is denoted by $d\Sigma_t$.

Suppose $U \subset \Sigma$ is an open subset. We consider the variation of the volume functional

$$\mathcal{A}(t) = |\iota_t(U)| = \int_U d\Sigma_t.$$

Proposition 1.3.1

The first variation of the volume functional is given by

$$\delta_V(U) := \mathcal{A}'(0) = \int_\Sigma \operatorname{div}^\Sigma V \, d\Sigma.$$

The second variation of the volume functional is given by

$$\begin{aligned}\delta_V^2(U) := \mathcal{A}''(0) = \int_\Sigma \sum_{i=1}^n -\langle R(V, e_i)V, e_i\rangle + \operatorname{div}^\Sigma \nabla_V V + (\operatorname{div}^\Sigma V)^2 \\ + \sum_{i=1}^n |(\nabla_{e_i} V)^\perp|^2 - \sum_{i,j=1}^n \langle \nabla_{e_i} V, e_j\rangle \langle \nabla_{e_j} V, e_i\rangle \, d\Sigma.\end{aligned}$$

Remark 1.3.2. Note that we do not impose any conditions on V and Σ for the above formulas. In particular, Σ does not need to be a critical point of the volume functional, and V does not need to be a normal variation.

Suppose $\{e_i\}$ is an orthonormal basis of $T\Sigma$, and define $e_i(t) = \iota_{t*} e_i$. Then the area element can be expressed as

$$d\Sigma_t = \sqrt{\det(\langle e_i(t), e_j(t)\rangle)} \, d\Sigma,$$

where $d\Sigma$ is the area element of Σ. We need the following lemma:

Lemma 1.3.3

We have

$$\det(I+tA) = 1 + t\cdot \operatorname{tr}(A) + \frac{t^2}{2}(\operatorname{tr}(A)^2 - \operatorname{tr}(A^2)) + O(t^3),$$

♠

Proof We recall the standard formula for the determinant of a matrix M:

$$\det(M) = \exp(\operatorname{tr}(\log M)).$$

Applying this to $I+tA$, we have

$$\begin{aligned}\det(I+tA) &= \exp(\operatorname{tr}(\log(I+tA)))\\ &= \exp\left(\operatorname{tr}\left(tA - \frac{t^2}{2}A^2 + O(t^3)\right)\right)\\ &= \exp\left(t\cdot\operatorname{tr}(A) - \frac{t^2}{2}\operatorname{tr}(A^2) + O(t^3)\right)\\ &= 1 + t\operatorname{tr}(A) + \frac{t^2}{2}(\operatorname{tr}(A)^2 - \operatorname{tr}(A^2)) + O(t^3).\end{aligned}$$

♦

We thus obtain the first variation formula:

$$\begin{aligned}&\frac{d}{dt}|_{t=0}|\Sigma_t\cap U| = \frac{d}{dt}|_{t=0}\int_\Sigma \sqrt{\det(\langle e_i, e_j\rangle)}\, d\Sigma\\ &= \frac{1}{2}\int_\Sigma \operatorname{tr}(\langle e_i', e_j\rangle + \langle e_i, e_j'\rangle)_{ij}\, d\Sigma = n\int_\Sigma H\, d\Sigma\\ &= \int_\Sigma \langle\nabla_{e_i}V, e_i\rangle\, d\Sigma = \int_\Sigma \operatorname{div}^\Sigma V\, d\Sigma\end{aligned}$$

For the second derivative of the area element, set $M(t) = (\langle e_i(t), e_j(t)\rangle)_{ij}$. Then

$$\begin{aligned}&\frac{d^2}{dt^2}|_{t=0}\sqrt{\det(M(t))} = \frac{1}{2}\frac{(\det M(t))''}{\sqrt{\det M(t)}} - \frac{1}{4}\frac{(\det M(t))'^2}{(\det M(t))^{3/2}}\\ &= \frac{1}{2}(\operatorname{tr} M'' + (\operatorname{tr} M')^2 - \operatorname{tr}(M'^2)) - \frac{1}{4}(\operatorname{tr} M')^2\\ &= \frac{1}{2}\operatorname{tr} M'' + \frac{1}{4}(\operatorname{tr} M')^2 - \frac{1}{2}\operatorname{tr}(M'^2).\end{aligned}$$

We compute each term:

$$\begin{aligned}\frac{1}{2}\operatorname{tr} M'' &= \sum_{i=1}^n \langle\nabla_V\nabla_V e_i, e_i\rangle + \sum_{i=1}^n\langle\nabla_V e_i, \nabla_V e_i\rangle\\ &= \sum_{i=1}^n -\langle R(V,e_i)V, e_i\rangle + \langle\nabla_{e_i}\nabla_V V, e_i\rangle + \sum_{i=1}^n\langle\nabla_{e_i}V, \nabla_{e_i}V\rangle\\ &= \sum_{i=1}^n -\langle R(V,e_i)V, e_i\rangle + \operatorname{div}^\Sigma\nabla_V V + \sum_{i=1}^n|\nabla_{e_i}V|^2,\end{aligned}$$

$$\frac{1}{4}(\operatorname{tr} M')^2 = \frac{1}{4}\left(\sum_{i=1}^n\langle\nabla_V e_i, e_i\rangle + \langle e_i, \nabla_V e_i\rangle\right)^2 = \left(\sum_{i=1}^n\langle\nabla_{e_i}V, e_i\rangle\right)^2 = (\operatorname{div}^\Sigma V)^2,$$

$$\begin{aligned}\frac{1}{2}\operatorname{tr}(M'^2) &= \frac{1}{2}\sum_{i,j=1}^n(\langle\nabla_{e_i}V, e_j\rangle + \langle\nabla_{e_j}V, e_i\rangle)^2 = \sum_{i,j=1}^n\langle\nabla_{e_i}V, e_j\rangle^2 + \sum_{i,j=1}^n\langle\nabla_{e_i}V, e_j\rangle\langle\nabla_{e_j}V, e_i\rangle\\ &= \sum_i|(\nabla_{e_i}V)^\top|^2 + \sum_{i,j=1}^n\langle\nabla_{e_i}V, e_j\rangle\langle\nabla_{e_j}V, e_i\rangle.\end{aligned}$$

So

$$\frac{d^2}{dt^2}|_{t=0}\sqrt{\det(M(t))} = \sum_{i=1}^{n} -\langle R(V,e_i)V,e_i\rangle + \operatorname{div}^\Sigma \nabla_V V + (\operatorname{div}^\Sigma V)^2$$
$$+\sum_{i=1}^{n}|(\nabla_{e_i}V)^\perp|^2 - \sum_{i,j=1}^{n}\langle \nabla_{e_i}V,e_j\rangle \left\langle \nabla_{e_j}V,e_i\right\rangle .$$

Then, we have the second variation formula:

$$\frac{d^2}{dt^2}|_{t=0}|\Sigma_t \cap U| = \int_\Sigma \frac{d^2}{dt^2}|_{t=0}\sqrt{\det(M(t))}\, d\Sigma$$
$$= \int_\Sigma \sum_{i=1}^{n} -\langle R(V,e_i)V,e_i\rangle + \operatorname{div}^\Sigma \nabla_V V + (\operatorname{div}^\Sigma V)^2$$
$$+\sum_{i=1}^{n}|(\nabla_{e_i}V)^\perp|^2 - \sum_{i,j=1}^{n}\langle \nabla_{e_i}V,e_j\rangle \left\langle \nabla_{e_j}V,e_i\right\rangle \, d\Sigma.$$

If V has compact support in U, then by the divergence theorem

$$\int_\Sigma \operatorname{div}^\Sigma V\, d\Sigma = \int_\Sigma \operatorname{div}^\Sigma (V^\perp + V^\top) d\Sigma$$
$$= \int_\Sigma \operatorname{div}^\Sigma V^\perp d\Sigma = \int_\Sigma \left\langle \vec{H}, V\right\rangle d\Sigma.$$

Definition 1.3.4 (Minimal submanifold)

A submanifold Σ is called ***minimal*** *if it is a critical point of the volume functional, i.e., $\delta_V(U) = 0$ for all variations V with compact support in U.* ♣

By the first variation formula, Σ is minimal if and only if $\vec{H} = 0$.

Suppose Σ is minimal, and the variation vector field V is normal, i.e., $V = V^\perp$, with compact support in U. Then, we know $\operatorname{div}^\Sigma V = 0$, and $\int_\Sigma \operatorname{div}^\Sigma \nabla_V V\, d\Sigma = 0$. So the second variation formula simplifies to

$$\left.\frac{d^2}{dt^2}\right|_{t=0}|\Sigma_t \cap U| = \int_\Sigma \sum_{i=1}^{n} -\langle R(V,e_i)V,e_i\rangle + \sum_{i=1}^{n}|(\nabla_{e_i}V)^\perp|^2 - \sum_{i,j=1}^{n}\langle \nabla_{e_i}V,e_j\rangle \left\langle \nabla_{e_j}V,e_i\right\rangle \, d\Sigma.$$

Furthermore, if Σ is a two-sided hypersurface, then we can write $V = f\nu$ for some smooth function f with compact support in U. In this case, the second variation formula becomes

$$\left.\frac{d^2}{dt^2}\right|_{t=0}|\Sigma_t \cap U| = \int_\Sigma |\nabla f|^2 - (|A|^2 + \operatorname{Ric}(\nu,\nu))f^2\, d\Sigma,$$

This formula is useful in the study of minimal hypersurfaces. For instance, it directly yields the second variation formula for minimal hypersurfaces with free boundary.

Proposition 1.3.5

Suppose $(\Sigma, \partial\Sigma) \hookrightarrow (M, \partial M)$ is a minimal hypersurface with free boundary, i.e., Σ meets ∂M orthogonally along $\partial\Sigma$. Then the second variation formula for normal variations $V = f\nu$ with compact support in U is

$$\left.\frac{d^2}{dt^2}\right|_{t=0}|\Sigma_t \cap U| = \int_\Sigma |\nabla f|^2 - (|A|^2 + \operatorname{Ric}(\nu,\nu))f^2\, d\Sigma - \int_{\partial\Sigma} A_{\partial M}(\nu,\nu)f^2\, d\sigma,$$

♠

Proof It remains only to handle the term $\int_\Sigma \operatorname{div}^\Sigma \nabla_V V\, d\Sigma$ in the second variation formula. By the divergence theorem, we have

$$\int_\Sigma \operatorname{div}^\Sigma \nabla_V V\, d\Sigma = \int_{\partial\Sigma} \langle \nabla_V V, \eta\rangle\, d\sigma = \int_{\partial\Sigma} A_{\partial M}(\nu,\nu)f^2\, d\sigma,$$

◆

Definition 1.3.6 (Stable minimal hypersurface)

A minimal hypersurface Σ is called ***stable*** *if the second variation of the volume functional is nonnegative for all variations with compact support, i.e.,* $\frac{d^2}{dt^2}\Big|_{t=0} |\Sigma_t \cap U| \geq 0$ *whenever* $\operatorname{spt} V \subset U$.

♣

If Σ is a two-sided stable minimal hypersurface, this is equivalent to the following stability inequality for all smooth functions f with compact support in U:

$$\int_\Sigma |\nabla f|^2 - (|A|^2 + \operatorname{Ric}(\nu,\nu)) f^2 \, d\Sigma \geq 0.$$

Corollary 1.3.7

The variation of mean curvature H is given by

$$\frac{d}{dt} H = -\Delta f - (|A|^2 + \operatorname{Ric}(\nu,\nu)) f,$$

where $V = f\nu$ is a normal variation with compact support in U.

♠

Proof We consider only the normal variation $V = f\nu$. The first variation gives

$$\int_\Sigma H f d\Sigma.$$

Hence the second variation can be expressed as

$$\int_\Sigma H \frac{d}{dt} f + \frac{d}{dt} H f + (Hf)^2 d\Sigma.$$

Comparing this expression with the second variation formula, we conclude that

$$\begin{aligned}
\int_\Sigma \frac{d}{dt} H f &= \int_\Sigma - \sum_{i=1}^n \langle R(V, e_i)V, e_i \rangle + \sum_{i=1}^n |(\nabla_{e_i} V)^\perp|^2 - \sum_{i,j=1}^n \langle \nabla_{e_i} V, e_j \rangle \langle \nabla_{e_j} V, e_i \rangle \\
&= \int_\Sigma |\nabla f|^2 - (|A|^2 + \operatorname{Ric}(\nu,\nu)) f^2 d\Sigma \\
&= \int_\Sigma -f \left(\Delta f + (|A|^2 + \operatorname{Ric}(\nu,\nu)) f \right) d\Sigma
\end{aligned}$$

Since this holds for all f with compact support in U, we have

$$\frac{d}{dt} H = -\Delta f - (|A|^2 + \operatorname{Ric}(\nu,\nu)) f.$$

◆

Define the Jacobi operator L by

$$L = -\Delta - (|A|^2 + \operatorname{Ric}(\nu,\nu)).$$

Then the stability condition can be written as

$$\int_\Sigma f L f \, d\Sigma \geq 0 \quad \text{for all } f \text{ with compact support in } U.$$

Definition 1.3.8 (Morse index)

The ***Morse index*** *of a minimal hypersurface Σ is the dimension of the space of smooth functions f with compact support in U such that*

$$\int_\Sigma f L f \, d\Sigma < 0,$$

i.e., the maximum dimension of a subspace on which the quadratic form is negative definite. ♣

Remark 1.3.9. Equivalently, the Morse index counts the number of negative eigenvalues (with multiplicity) of the Jacobi operator L under appropriate boundary conditions. A minimal hypersurface is stable if and only if its Morse index is zero.

Theorem 1.3.10

Let Σ^n be a complete minimal hypersurface in a Riemannian manifold (M^{n+1}, g). If Σ has finite Morse index, then Σ is stable outside a compact set, i.e., there exists a compact set $K \subset \Sigma$ such that

$$\int_\Sigma |\nabla f|^2 - (|A|^2 + \mathrm{Ric}(\nu,\nu)) f^2 \, d\Sigma \geq 0$$

for all $f \in C_c^\infty(\Sigma \setminus K)$. ♡

Proof Let $\mathrm{ind}(\Sigma) = m < \infty$ be the Morse index of Σ. By definition, there exists a finite-dimensional subspace $V \subset C_c^\infty(\Sigma)$ of dimension m such that the quadratic form

$$Q(f) = \int_\Sigma |\nabla f|^2 - (|A|^2 + \mathrm{Ric}(\nu,\nu)) f^2 \, d\Sigma$$

is negative definite on V, and $Q(f) \geq 0$ for all $f \perp V$ (in the L^2 inner product sense).

Since each function f_i in a basis of V has compact support, there exists a compact set $K \subset \Sigma$ such that $\mathrm{spt}(f_i) \subset K$ for all $i = 1, \ldots, m$.

Now, for any $f \in C_c^\infty(\Sigma \setminus K)$, we have $\mathrm{spt}(f) \cap K = \emptyset$. This means f is orthogonal to every basis function f_i of V (since their supports are disjoint). Therefore, $f \perp V$, and by the definition of finite Morse index, we have

$$Q(f) = \int_\Sigma |\nabla f|^2 - (|A|^2 + \mathrm{Ric}(\nu,\nu)) f^2 \, d\Sigma \geq 0.$$

This proves that Σ is stable outside the compact set K. ♦

1.4 Properties of Minimal Submanifolds in $\mathbb{R}^N$

Proposition 1.4.1

Let Σ^n be a submanifold in $\mathbb{R}^N$ with position vector $x = (x_1, \ldots, x_N)$. Then its mean curvature vector $\vec{H}$ is given by

$$\Delta^\Sigma x = \vec{H}.$$

♠

Proof Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $T\Sigma$. The Laplacian of the coordinate x_k on Σ is given by

$$\Delta^\Sigma x_k = \sum_{i=1}^n D_{e_i} D_{e_i} x_k = \sum_{i=1}^n D_{e_i}(e_i \cdot \frac{\partial}{\partial x_k}) = (D_{e_i} e_i) \cdot \frac{\partial}{\partial x_k} = \vec{H} \cdot \frac{\partial}{\partial x_k},$$

Hence

$$\Delta^\Sigma x = \vec{H}.$$

♦

Proposition 1.4.2

Let Σ^n be a submanifold in $\mathbb{R}^N$ with position vector $x = (x_1, \dots, x_N)$. Then Σ is minimal if and only if each coordinate function x_i is harmonic on Σ, i.e., $\Delta^\Sigma x_i = 0$ for all $i = 1, \dots, N$.

Proposition 1.4.3

Suppose Σ^n is a minimal submanifold in $\mathbb{R}^{n+1}$. Then $\nu \cdot \frac{\partial}{\partial x_i}$ satisfies the following Jacobi equation:

$$\Delta^\Sigma(\nu \cdot \frac{\partial}{\partial x_i}) + |A|^2(\nu \cdot \frac{\partial}{\partial x_i}) = 0.$$

We may write this more concisely as

$$L\nu = 0.$$

Proof This is because the flow generated by $\frac{\partial}{\partial x_i}$ is a family of isometries of $\mathbb{R}^{n+1}$, so $\frac{d}{dt}H = 0$ for the corresponding variation. By the variation formula for the mean curvature, we have

$$0 = \frac{d}{dt}H = -\Delta^\Sigma(\nu \cdot \frac{\partial}{\partial x_i}) - |A|^2(\nu \cdot \frac{\partial}{\partial x_i}).$$

1.5 Monotonicity Formula

Proposition 1.5.1

Suppose Σ^n is a minimal submanifold in $\mathbb{R}^N$ and $x_0 \in \mathbb{R}^N$. Then we have the following monotonicity formula:

$$\frac{|\Sigma \cap B_\rho(x_0)|}{\rho^n} - \frac{|\Sigma \cap B_\sigma(x_0)|}{\sigma^n} = \int_{\Sigma \cap (B_\rho(x_0) \setminus B_\sigma(x_0))} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^{n+2}} d\Sigma$$

Proof Assume $x_0 = 0$, and choose the following vector field:

$$V = \begin{cases} x\left(\frac{1}{|x|^n} - \frac{1}{\rho^n}\right), & \sigma \le |x| \le \rho \\ x\left(\frac{1}{\sigma^n} - \frac{1}{\rho^n}\right), & |x| < \sigma \end{cases}$$

So

$$\begin{aligned} 0 = \int_\Sigma \operatorname{div}^\Sigma V \, d\Sigma &= \int_{B_\sigma} \frac{n}{\sigma^n} d\Sigma - \int_\Sigma \frac{n}{\rho^n} d\Sigma + \int_{B_\rho \setminus B_\sigma} \operatorname{div}^\Sigma \frac{x}{|x|^n} d\Sigma \\ &= \frac{n|\Sigma \cap B_\sigma|}{\sigma^n} - \frac{n|\Sigma \cap B_\rho|}{\rho^n} + \int_{B_\rho \setminus B_\sigma} \frac{n}{|x|^n} - \frac{n|x^T|^2}{|x|^{n+2}} d\Sigma \\ &= \frac{n|\Sigma \cap B_\sigma|}{\sigma^n} - \frac{n|\Sigma \cap B_\rho|}{\rho^n} + \int_{B_\rho \setminus B_\sigma} \frac{n|x^\perp|^2}{|x|^{n+2}} d\Sigma. \end{aligned}$$

1.6 Area-Minimizing Hypersurfaces and Calibration

Definition 1.6.1 (Area-minimizing hypersurface)

A (complete) hypersurface Σ^n in a Riemannian manifold (M^N, g) is called (absolutely) ***area-minimizing*** *if it minimizes the area functional among all hypersurfaces agreeing with Σ outside a compact set. More precisely, Σ is area-minimizing if for every compact set $K \subset M$, we have*

$$|\Sigma \cap K| \leq |\Sigma' \cap K|$$

for all hypersurfaces Σ' with $\Sigma' \setminus K = \Sigma \setminus K$.

We say Σ is ***area-minimizing in its homology class*** *if we also require Σ' to be homologous to Σ, i.e., $\Sigma - \Sigma' = \partial\Gamma$ for some $(n+1)$-dimensional chain Γ in M.*

One can also define area-minimizing submanifolds in other classes, e.g., homotopy classes. ♣

Proposition 1.6.2

Every area-minimizing hypersurface is minimal, i.e., its mean curvature vector $\vec{H} = 0$. ♠

Proposition 1.6.3

Area-minimizing hypersurfaces are stable. That is, the second variation of the area functional is non-negative for all variations with compact support. ♠

Definition 1.6.4 (Calibration)

Let (M, g) be a Riemannian manifold. A k-form $\omega \in \Omega^k(M)$ is called a ***calibration*** *if it satisfies:*

1. ***Closedness:*** $d\omega = 0$.
2. ***Comass*** ≤ 1***:*** *For every point $p \in M$ and every unit simple k-vector ξ at p, we have*

$$\omega_p(\xi) \leq 1.$$

♣

Definition 1.6.5 (Calibrated submanifold)

An oriented k-dimensional submanifold Σ^k in M is ***calibrated*** *by a k-form ω if $\iota^*\omega = d\mathcal{H}^k|_\Sigma$, where $\iota : \Sigma \hookrightarrow M$ is the inclusion map and $d\mathcal{H}^k|_\Sigma$ is the volume form on Σ induced by the Riemannian metric.* ♣

Theorem 1.6.6

If an oriented k-dimensional submanifold Σ^k is calibrated by a closed k-form ω, then Σ is area-minimizing in its homology class. More precisely, for any other oriented k-dimensional submanifold Σ' with $\partial\Sigma' = \partial\Sigma$, we have

$$|\Sigma| \leq |\Sigma'|.$$

♡

Proof For simplicity, we assume Σ is a compact submanifold with boundary. Let Σ' be any k-dimensional oriented surface with $\partial\Sigma' = \partial\Sigma$. Define the $(k+1)$-dimensional chain Γ such that $\partial\Gamma = \Sigma - \Sigma'$.

So we have

$$\int_\Sigma \omega - \int_{\Sigma'} \omega = \int_\Gamma d\omega = 0,$$

since ω is closed. Now, we have

$$|\Sigma| = \int_\Sigma \omega = \int_{\Sigma'} \omega \leq |\Sigma'|,$$

which shows that Σ minimizes area among all surfaces with the same boundary. ◆

Theorem 1.6.7

Any complex analytic variety (i.e., complex submanifold or more generally, integral current defined by a holomorphic equation) in $\mathbb{C}^2$ is absolutely area-minimizing. ♡

Proof The proof uses the theory of calibrations. In $\mathbb{C}^2$, consider the standard Kähler form

$$\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) = dx_1 \wedge dy_1 + dx_2 \wedge dy_2,$$

The real 2-form ω is closed ($d\omega = 0$) and has comass 1, i.e., for any oriented 2-plane ξ in $\mathbb{C}^2$, $\omega|_\xi \leq 1$ with equality if and only if ξ is a complex line.

Any complex curve (complex 1-dimensional submanifold) $\Sigma \subset \mathbb{C}^2$ is calibrated by ω, since the restriction of ω to Σ is exactly the area form of Σ:

$$\omega|_\Sigma = d\mathcal{H}^2|_\Sigma.$$

By the preceding calibration argument, Σ is area-minimizing in its homology class. Also note that $\mathbb{R}^4$ has trivial second homology group, so any two surfaces with the same boundary are automatically homologous.

Therefore, any complex analytic variety in $\mathbb{C}^2$ is absolutely area-minimizing. ◆

Example 1.1 (Singular complex curve). The set $\{z^2 = w^3\}$ is a complex analytic variety in $\mathbb{C}^2$ with an isolated singularity at the origin. It is area-minimizing, but not smooth.

1.7 Minimal Graphs

Definition 1.7.1 (Minimal graph)

Let $\Omega \subset \mathbb{R}^n$ be an open domain and $u : \Omega \to \mathbb{R}$ be a smooth function. The ***graph*** *of u is the submanifold*

$$\Sigma_u = \{(x, u(x)) : x \in \Omega\} \subset \mathbb{R}^{n+1}.$$

We say Σ_u is a ***minimal graph*** *if it is a minimal hypersurface in $\mathbb{R}^{n+1}$, i.e., its mean curvature vanishes identically.* ♣

The graph Σ_u is parametrized by the immersion $\iota : \Omega \to \mathbb{R}^{n+1}$ given by $\iota(x) = (x, u(x))$. The tangent vectors are

$$\eta_i = \partial_i \iota = \partial_i + \partial_i u\, \partial_{n+1}, \quad i = 1, \ldots, n,$$

where $\{\partial_1, \ldots, \partial_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$.

Basic geometry of graphs.

The induced metric on Σ_u is given by

$$g_{ij} = \delta_{ij} + \partial_i u\, \partial_j u,$$

and its determinant is

$$\det(g_{ij}) = 1 + |\nabla u|^2,$$

where $|\nabla u|^2 = \sum_{i=1}^n (\partial_i u)^2$. Thus, the area functional of the graph over Ω is

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx.$$

The upward-pointing unit normal to Σ_u is

$$\nu = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}} = \frac{1}{W}(-\partial_1 u, \dots, -\partial_n u, 1),$$

where we denote $W = \sqrt{1+|\nabla u|^2}$.

The second fundamental form of Σ_u is computed by

$$A_{ij} = -\langle \partial_i \nu, \partial_j \iota \rangle = \frac{\partial_i \partial_j u}{W} = \frac{u_{ij}}{W}.$$

The inverse of the induced metric is

$$g^{ij} = \delta^{ij} - \frac{\partial_i u\, \partial_j u}{W^2}.$$

Proposition 1.7.2

The mean curvature of the graph Σ_u is given by

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \sum_{i=1}^{n} \partial_i \left(\frac{\partial_i u}{\sqrt{1+|\nabla u|^2}}\right).$$

♠

Proof The mean curvature is the trace of the second fundamental form with respect to the induced metric:

$$\begin{aligned} H = g^{ij} A_{ij} &= \left(\delta^{ij} - \frac{\partial_i u\, \partial_j u}{W^2}\right) \frac{u_{ij}}{W} \\ &= \frac{1}{W}\left(\Delta u - \frac{\partial_i u\, \partial_j u\, u_{ij}}{W^2}\right) \\ &= \frac{1}{W}\Delta u - \frac{\partial_i u\, \partial_j u\, u_{ij}}{W^3} = \operatorname{div}\left(\frac{\nabla u}{W}\right). \end{aligned}$$

♦

Definition 1.7.3 (Minimal surface equation)

The ***minimal surface equation*** *(MSE) is the quasilinear elliptic PDE*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0,$$

or equivalently,

$$(1+|\nabla u|^2)\Delta u - \sum_{i,j=1}^{n} \partial_i u\, \partial_j u\, \partial_i \partial_j u = 0.$$

♣

The minimal surface equation is the Euler–Lagrange equation of the area functional

$$\mathcal{A}(u) = \int_\Omega \sqrt{1+|\nabla u|^2}\, dx.$$

Indeed, for any compactly supported variation $\phi \in C_c^\infty(\Omega)$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(u+t\phi) = \int_\Omega \frac{\nabla u \cdot \nabla \phi}{\sqrt{1+|\nabla u|^2}}\, dx = -\int_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \phi\, dx.$$

Hence, u is a critical point of $\mathcal{A}$ if and only if it satisfies the minimal surface equation.

Example 1.2 (Classical minimal graphs).

(i) **Scherk's surface.** In $\mathbb{R}^3$, the function

$$u(x_1, x_2) = \log\left(\frac{\cos x_1}{\cos x_2}\right)$$

defined on $\Omega = \{|x_1| < \pi/2\} \cap \{|x_2| < \pi/2\}$ is a solution of the minimal surface equation, known as

Scherk's first surface.

(ii) **Catenoid.** The catenoid is a minimal surface of revolution in $\mathbb{R}^3$ that can be locally written as a graph $u(r) = \cosh^{-1}(r)$ for $r \geq 1$.

1.8 The Dirichlet Problem for the Minimal Surface Equation

Definition 1.8.1 (Dirichlet problem for the minimal surface equation)

*The **Dirichlet problem** for the minimal surface equation asks: given a bounded domain $\Omega \subset \mathbb{R}^n$ and boundary data $\phi \in C^0(\partial\Omega)$, find $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that*

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases}$$

♣

Definition 1.8.2 (Mean-convex domain)

*A bounded C^2 domain $\Omega \subset \mathbb{R}^n$ is **mean convex** if the mean-curvature vector of $\partial\Omega$ points weakly into Ω. Equivalently, with the scalar convention for which Euclidean balls are mean convex, the boundary mean curvature satisfies $H_{\partial\Omega} \geq 0$.*

♣

Theorem 1.8.3 (Jenkins–Serrin [JS68])

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain. Then the Dirichlet problem for the minimal surface equation has a solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ for every boundary data $\phi \in C^0(\partial\Omega)$ if and only if Ω is mean convex. In this case the solution is unique. If, moreover, $\partial\Omega$ and ϕ are $C^{2,\alpha}$, then the solution is $C^{2,\alpha}$ up to the boundary by the standard boundary regularity theory for quasilinear elliptic equations.

♡

Thus mean convexity is part of the existence theorem, not merely a technical regularity assumption. On a non-mean-convex bounded domain, arbitrary boundary data need not be solvable; Jenkins–Serrin instead prove solvability under an additional smallness condition involving $\operatorname{osc}(\phi)$ and the first two boundary derivatives of ϕ.

Definition 1.8.4 (Direction field)

*For a minimal graph $\Sigma_u = \{(x, u(x)) : x \in \Omega\}$, the **direction field** is defined as*

$$X = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}}.$$

♣

This direction field satisfies

$$\operatorname{div} X = 0,$$
$$X \cdot \nu = 1.$$

where ν is the unit normal to Σ_u.

Theorem 1.8.5

Every minimal graph in $\mathbb{R}^{n+1}$ is area-minimizing within the class of hypersurfaces with the same boundary.

♡

Proof Define the vector field

$$V = \frac{(-\nabla u, 1)}{\sqrt{1+|\nabla u|^2}}.$$

Then the form $\omega = \iota_V d\mathrm{vol}$ is a calibration, and Σ_u is calibrated by ω. Therefore, Σ_u minimizes area among all hypersurfaces with the same boundary. ◆

Corollary 1.8.6

Every minimal graph in $\mathbb{R}^{n+1}$ is stable. ♠

1.9 Bernstein's Theorem and Generalizations

Theorem 1.9.1 (Bernstein's Theorem [Ber17])

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be an entire solution of the minimal surface equation. Then u is an affine function, i.e., $u(x_1, x_2) = ax_1 + bx_2 + c$ for some constants $a, b, c \in \mathbb{R}$. ♡

This result was generalized to higher dimensions:

Theorem 1.9.2 (Fleming [Fle62]–De Giorgi [DG65]–Almgren [Alm66]–Simons [Sim68])

Let $u : \mathbb{R}^n \to \mathbb{R}$ be an entire solution of the minimal surface equation.

(i) If $n \leq 7$, then u must be affine.

(ii) For $n \geq 8$, there exist non-affine entire solutions (as shown by a counterexample due to Bombieri–De Giorgi–Giusti [BDGG69]). ♡

Proof of Bernstein's theorem in dimension 2 Let

$$\Sigma = \{(x, u(x)) : x \in \mathbb{R}^2\} \subset \mathbb{R}^3$$

be the graph of an entire solution of the minimal surface equation. By the calibration argument above, Σ is area-minimizing. In particular, Σ is stable, so for every $\varphi \in C_c^\infty(\Sigma)$,

$$\int_\Sigma |A|^2 \varphi^2 \, d\Sigma \leq \int_\Sigma |\nabla^\Sigma \varphi|^2 \, d\Sigma,$$

since $\mathrm{Ric}_{\mathbb{R}^3} = 0$.

We first establish a quadratic area bound. For a.e. $R > 0$, the intersection

$$\Gamma_R := \Sigma \cap \partial B_R$$

is a smooth 1-cycle in the sphere $\partial B_R = S_R$. Since $H_1(S_R) = 0$, the curve Γ_R bounds a region $D_R \subset S_R$. Replacing D_R by its complement if necessary, we may assume

$$|D_R| \leq \frac{1}{2}|S_R| = 2\pi R^2.$$

Since Σ is area-minimizing and $\partial(\Sigma \cap B_R) = \Gamma_R = \partial D_R$, we obtain

$$|\Sigma \cap B_R| \leq |D_R| \leq 2\pi R^2$$

for a.e. $R > 0$.

Now choose the logarithmic cutoff

$$\eta_R(x) = \begin{cases} 1, & |x| \le R, \\ \dfrac{k + \log R - \log|x|}{k}, & R < |x| < e^k R, \\ 0, & |x| \ge e^k R, \end{cases}$$

Here $|x|$ denotes the Euclidean distance to the origin in $\mathbb{R}^3$. Since $|\nabla^\Sigma |x|| \le 1$, on the annulus $R < |x| < e^k R$ we have

$$|\nabla^\Sigma \eta_R| \le \frac{1}{k|x|}.$$

Applying stability with $\varphi = \eta_R$ gives

$$\int_\Sigma |A|^2 \eta_R^2 \, d\Sigma \le \int_\Sigma |\nabla^\Sigma \eta_R|^2 \, d\Sigma.$$

To estimate the right-hand side, decompose

$$B_{e^k R} \setminus B_R = \bigcup_{i=0}^{k-1} \big(B_{e^{i+1}R} \setminus B_{e^i R}\big),$$

Using the area bound,

$$\begin{aligned} \int_\Sigma |\nabla^\Sigma \eta_R|^2 \, d\Sigma &\le \frac{1}{k^2} \sum_{i=0}^{k-1} \int_{\Sigma \cap (B_{e^{i+1}R} \setminus B_{e^i R})} \frac{1}{|x|^2} \, d\Sigma \\ &\le \frac{1}{k^2} \sum_{i=0}^{k-1} \frac{1}{(e^i R)^2} \, |\Sigma \cap B_{e^{i+1}R}| \\ &\le \frac{1}{k^2} \sum_{i=0}^{k-1} \frac{1}{(e^i R)^2} \, 2\pi (e^{i+1}R)^2 \\ &\le \frac{C}{k}. \end{aligned}$$

Hence

$$\int_{\Sigma \cap B_R} |A|^2 \, d\Sigma \le \frac{C}{k}.$$

Letting $k \to \infty$, we conclude that $A \equiv 0$ on $\Sigma \cap B_R$. Since this holds for a.e. $R > 0$, we have $A \equiv 0$ on Σ. Therefore Σ is totally geodesic, hence a plane in $\mathbb{R}^3$.

Since Σ is a graph over $\mathbb{R}^2$, that plane cannot be vertical. Thus

$$u(x_1, x_2) = ax_1 + bx_2 + c$$

for some constants $a, b, c \in \mathbb{R}$. This proves the theorem. ◆

Remark 1.9.3. The key point in dimension 2 is that stability plus the quadratic area growth

$$|\Sigma \cap B_R| \le CR^2$$

allows the logarithmic cutoff to force

$$\int_\Sigma |A|^2 \, d\Sigma = 0.$$

This argument is specific to two dimensions and yields the classical stability proof of Bernstein's theorem.

Chapter 2 Simons' Inequality and Generalized Bernstein Theorems

We now discuss several generalizations of Bernstein's theorem and curvature estimates for stable minimal hypersurfaces.

2.1 Simons' Identity and Inequality

Lemma 2.1.1 (Simons' Identity [Sim68])

For a minimal hypersurface Σ^n in $\mathbb{R}^{n+1}$, the second fundamental form A satisfies the following identity:

$$\frac{1}{2}\Delta|A|^2 = -|A|^4 + |\nabla A|^2.$$

Proof

Recall that the Ricci identity states that for any 2-tensor T_{ij} on Σ, we have

$$T_{ij,kl} - T_{ij,lk} = R_{lkim}T_{mj} + R_{lkjm}T_{im},$$

where R_{ijkl} is the Riemann curvature tensor of Σ. The Gauss equation in Euclidean space gives

$$R_{ijkl} = A_{ik}A_{jl} - A_{il}A_{jk}.$$

Hence

$$\frac{1}{2}\Delta|A|^2 = A_{ij}\Delta A_{ij} + A_{ij,k}^2$$

Moreover,

$$\begin{aligned}
A_{ij}\Delta A_{ij} &= A_{ij}A_{ij,kk} = A_{ij}A_{ik,jk} \quad \text{(Codazzi equation)}\\
&= A_{ij}A_{ik,kj} + A_{ij}(R_{kjim}A_{mk} + R_{kjkm}A_{im}) \quad \text{(Ricci identity)}\\
&= A_{ij}A_{kk,ij} + A_{ij}(A_{ki}A_{jm} - A_{ji}A_{km})A_{mk}\\
&\quad + A_{ij}(A_{kk}A_{jm} - A_{km}A_{jk})A_{im} \quad \text{(Gauss equation, Codazzi equation)}\\
&= -A_{ij}A_{ji}A_{km}A_{mk} + A_{ij}A_{jm}A_{mk}A_{ki} - A_{ij}A_{jk}A_{km}A_{mi} \quad \text{minimal}\\
&= -|A|^4.
\end{aligned}$$

This proves Simons' identity.

Lemma 2.1.2 (Refined Kato inequality)

For the second fundamental form A of a minimal hypersurface Σ^n in $\mathbb{R}^{n+1}$, we have

$$|\nabla A|^2 \geq \left(1 + \frac{2}{n}\right)|\nabla|A||^2.$$

Proof Let $\{e_i\}$ be a local orthonormal frame on Σ that diagonalizes A at a point, so $A_{ij} = \lambda_i\delta_{ij}$ with $\sum_i \lambda_i = 0$. Then

$$|\nabla|A|^2|^2 = \left|\nabla\sum_i \lambda_i^2\right|^2 = \left(2\sum_i \lambda_i A_{ii,k}\right)^2.$$

Hence

$$|\nabla |A|^2|^2 = 4\sum_k \Big(\sum_i \lambda_i A_{ii,k}\Big)^2 \le 4\sum_i \lambda_i^2 \sum_{i,k} A_{ii,k}^2 = 4|A|^2 \sum_{i,k} A_{ii,k}^2.$$

Since $2|A|\,|\nabla |A|| = |\nabla |A|^2|$, we have

$$|\nabla |A||^2 = \frac{|\nabla |A|^2|^2}{4|A|^2} \le \sum_{i,k} A_{ii,k}^2.$$

Moreover,

$$\begin{aligned}
\sum_{i,k=1}^{n} A_{ii,k}^2 &= \sum_{i\neq k} A_{ii,k}^2 + \sum_{i=1}^{n} A_{ii,i}^2 \\
&\le \sum_{i\neq k} A_{ii,k}^2 + \sum_{i=1}^{n} (\sum_{j\neq i} A_{jj,i})^2 \\
&\le \sum_{i\neq k} A_{ii,k}^2 + (n-1)\sum_{i=1}^{n}\sum_{j\neq i} A_{jj,i}^2 \\
&= n\sum_{i\neq k} A_{ii,k}^2 \\
&= \frac{n}{2}\sum_{i\neq j,k} A_{ik,i}^2 + A_{ki,i}^2
\end{aligned}$$

Therefore,

$$\left(1+\frac{2}{n}\right)|\nabla |A||^2 \le \sum_{i,k} A_{ii,k}^2 + \sum_{i\neq j,k} A_{ik,i}^2 + A_{ki,i}^2 \le |\nabla A|^2.$$

◆

Corollary 2.1.3

For a minimal hypersurface Σ^n in $\mathbb{R}^{n+1}$, the second fundamental form A satisfies the following differential inequality:

$$\Delta |A|^2 \ge -2|A|^4 + 2\left(1+\frac{2}{n}\right)|\nabla |A||^2.$$

Equivalently,

$$|A|\Delta |A| + |A|^4 \ge \frac{2}{n}|\nabla |A||^2.$$

♠

Theorem 2.1.4 (Schoen–Simon–Yau [SSY75])

Let Σ^n be an (immersed) stable minimal hypersurface in $\mathbb{R}^{n+1}$ with $n \le 5$. Suppose Σ has (intrinsic) Euclidean volume growth, i.e., there exists a constant $C > 0$ such that for all $R > 0$,

$$|B_R^\Sigma| \le CR^n,$$

where B_R^Σ is the intrinsic geodesic ball of radius R in Σ. Then Σ must be a hyperplane.

♡

Proof We test the stability inequality with $|A|^{p-1}\varphi$. Then

$$\begin{aligned}
\int |A|^{2p}\varphi^2 &\le \int |\nabla(|A|^{p-1}\varphi)|^2 \\
&= \int (p-1)^2|A|^{2p-4}|\nabla |A||^2\varphi^2 + |A|^{2p-2}|\nabla\varphi|^2 + (2p-2)|A|^{2p-3}\varphi\langle\nabla |A|, \nabla\varphi\rangle.
\end{aligned}$$

We multiply Simons' inequality by $|A|^{2p-4}\varphi^2$ and integrate by parts to obtain

$$\frac{2}{n}\int |A|^{2p-4}|\nabla|A||^2\varphi^2 \le \int |A|^{2p}\varphi^2 - (2p-3)|\nabla|A||^2|A|^{2p-4}\varphi^2 - 2|A|^{2p-3}\varphi\langle\nabla|A|,\nabla\varphi\rangle.$$

Cancelling the $|A|^{2p}\varphi^2$ term, we obtain

$$\left(\frac{2}{n}+(2p-3)-(p-1)^2\right)\int |A|^{2p-4}|\nabla|A||^2\varphi^2 \le \int |A|^{2p-2}|\nabla\varphi|^2 + (2p-4)|A|^{2p-3}\varphi|\nabla|A|||\nabla\varphi|.$$

If $\frac{2}{n}+(2p-3)-(p-1)^2>0$, i.e.,

$$(p-2)^2 < \frac{2}{n},$$

then we can apply Cauchy–Schwarz to the right-hand side to get

$$\int |A|^{2p-4}|\nabla|A||^2\varphi^2 \le C\int |A|^{2p-2}|\nabla\varphi|^2,$$

for some constant C depending on n and p. Using the stability inequality again, we get

$$\int |A|^{2p}\varphi^2 \le C\int |A|^{2p-2}|\nabla\varphi|^2 \le C\left(\int |A|^{2p}\varphi^2\right)^{\frac{p-1}{p}}\left(\int |\nabla\varphi|^{2p}\right)^{\frac{1}{p}}.$$

Hence,

$$\int |A|^{2p}\varphi^2 \le C\int |\nabla\varphi|^{2p}.$$

If we choose φ to be a cutoff function supported in B^{Σ}_{2R} with $|\nabla\varphi|\le\frac{C}{R}$, and equal to 1 in B^{Σ}_{R}, then

$$\int_{B^{\Sigma}_R}|A|^{2p} \le C\int_{B^{\Sigma}_{2R}}\left(\frac{C}{R}\right)^{2p} \le CR^{n-2p}.$$

Here the parameter p is half of the final integrability exponent, since the estimate controls $|A|^{2p}$. Thus the decay requires $2p>n$, equivalently $p>n/2$. When $n\le 5$, the admissible interval

$$2-\sqrt{\frac{2}{n}} < p < 2+\sqrt{\frac{2}{n}}$$

intersects $(n/2,\infty)$. Choose such a p. Then $n-2p<0$, and letting $R\to\infty$ gives $A\equiv 0$ on Σ. Hence Σ is a hyperplane.

◆

Remark 2.1.5. Another useful hypothesis is extrinsic Euclidean volume growth: there exists $C>0$ such that for all $R>0$,

$$|\Sigma\cap B_R(0)|\le CR^n.$$

This condition implies intrinsic Euclidean volume growth.

Proposition 2.1.6

The following two statements are equivalent:

- *(i) Every complete stable minimal hypersurface in $\mathbb{R}^{n+1}$ with extrinsic Euclidean volume growth is a hyperplane. (This means $\Sigma\cap B_R(0)$ has area at most CR^n for some constant C independent of R.)*
- *(ii) We have the following curvature estimate for stable minimal hypersurfaces: there exists $C=C(n,\Lambda)$ such that if $\Sigma^n\subset B_{2R}(0)\subset\mathbb{R}^{n+1}$ is a stable minimal hypersurface with*

$$|\Sigma\cap B_{2R}(0)|\le\Lambda R^n,$$

then

$$\sup_{\Sigma\cap B_R(0)}|A|\le\frac{C}{R}.$$

♠

Remark 2.1.7. Note that (ii) is scale-invariant. If it holds for some $R > 0$, then it holds for all $R > 0$ by scaling. This is because for $\lambda > 0$, the second fundamental form of $\lambda\Sigma$ is $\lambda^{-1}A$, and the area of $\lambda\Sigma \cap B_{2\lambda R}(0)$ is $\lambda^n|\Sigma \cap B_{2R}(0)|$.

Proof **(ii) $\Rightarrow$ (i).** Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be complete, stable, and minimal, with extrinsic Euclidean volume growth:

$$|\Sigma \cap B_R(0)| \leq C_0 R^n \quad \forall R > 0.$$

Fix $p \in \Sigma$. For $R \geq |p|$, the ball $B_{2R}(p)$ is contained in $B_{3R}(0)$, and hence

$$|\Sigma \cap B_{2R}(p)| \leq C_0(3R)^n.$$

Apply (ii) after translating p to the origin, with $\Lambda = 3^n C_0$:

$$\sup_{\Sigma \cap B_R(p)} |A| \leq \frac{C}{R}.$$

Let $R \to \infty$. Then $|A|(p) = 0$. Since p is arbitrary, $A \equiv 0$, so Σ is a hyperplane.

(i) $\Rightarrow$ (ii). Assume (ii) fails. Then for some $\Lambda > 0$ there exist stable minimal hypersurfaces

$$\Sigma_j \subset B_{2R_j}(0), \qquad |\Sigma_j \cap B_{2R_j}(0)| \leq \Lambda R_j^n,$$

such that

$$\sup_{\Sigma_j \cap B_{R_j}(0)} |A_j|\, R_j \to \infty.$$

Choose $y_j \in \Sigma_j \cap B_{R_j}(0)$ such that, with $K_j := |A_j|(y_j)$,

$$K_j R_j \to \infty.$$

We use the standard point-picking argument in the ball centered at y_j. Define

$$G_j(x) := \left(\frac{R_j}{2} - |x - y_j|\right) |A_j|(x)$$

on $\Sigma_j \cap B_{R_j/2}(y_j)$, and choose q_j where G_j attains its maximum. Since $G_j(y_j) = R_j K_j/2$, if

$$Q_j := |A_j|(q_j), \qquad \rho_j := \frac{1}{2}\left(\frac{R_j}{2} - |q_j - y_j|\right),$$

then

$$Q_j \rho_j = \frac{G_j(q_j)}{2} \geq \frac{R_j K_j}{4} \to \infty.$$

Moreover, for $x \in \Sigma_j \cap B_{\rho_j}(q_j)$,

$$\frac{R_j}{2} - |x - y_j| \geq \frac{1}{2}\left(\frac{R_j}{2} - |q_j - y_j|\right),$$

and the maximality of G_j gives

$$|A_j|(x) \leq 2Q_j.$$

Rescale:

$$\widetilde{\Sigma}_j := Q_j(\Sigma_j - q_j).$$

Then $0 \in \widetilde{\Sigma}_j$, $|\widetilde{A}_j|(0) = 1$, and on $B_{\widetilde{\rho}_j}(0)$ with $\widetilde{\rho}_j := Q_j \rho_j \to \infty$:

$$|\widetilde{A}_j| \leq 2.$$

Stability is scale-invariant, so each $\widetilde{\Sigma}_j$ is stable minimal.

We now get the local area bound in the rescaled sequence. Because $q_j \in B_{R_j/2}(y_j)$ and $y_j \in B_{R_j}(0)$, we have $q_j \in B_{3R_j/2}(0)$. Hence

$$B_{R_j/2}(q_j) \subset B_{2R_j}(0).$$

Fix $\sigma > 0$. Since $\rho_j \le R_j/4$ and $Q_j\rho_j \to \infty$, we also have $Q_jR_j \to \infty$. For j large, $\sigma/Q_j \le \rho_j$ and $\sigma/Q_j \le R_j/2$. By the monotonicity formula, applied with center q_j and the two radii σ/Q_j and $R_j/2$,

$$\frac{|\Sigma_j \cap B_{\sigma/Q_j}(q_j)|}{(\sigma/Q_j)^n} \le \frac{|\Sigma_j \cap B_{R_j/2}(q_j)|}{(R_j/2)^n} \le 2^n\Lambda.$$

Hence

$$|\widetilde{\Sigma}_j \cap B_\sigma(0)| = Q_j^n|\Sigma_j \cap B_{\sigma/Q_j}(q_j)| \le 2^n\Lambda\sigma^n.$$

Thus $\widetilde{\Sigma}_j$ have uniform local area growth and curvature bounds on larger and larger balls. By the compactness theorem for minimal immersions with locally bounded curvature and area, a subsequence converges smoothly on compact sets to a complete stable minimal hypersurface $\Sigma_\infty \subset \mathbb{R}^{n+1}$ with Euclidean volume growth. By construction,

$$|A_{\Sigma_\infty}|(0) = 1.$$

But (i) says every such Σ_∞ is a hyperplane, so $A_{\Sigma_\infty} \equiv 0$, contradiction.

Therefore (ii) must hold. ◆

Corollary 2.1.8

The curvature estimate in (ii) holds for $n \le 5$.

Theorem 2.1.9 (Bernstein theorem for $n \le 5$)

Let $u : \mathbb{R}^n \to \mathbb{R}$ be an entire solution of the minimal surface equation, with $n \le 5$. Then u is affine.

Proof Let

$$\Sigma = \{(x, u(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$$

be the graph of u. By the calibration argument above, Σ is area-minimizing. In particular, Σ is stable.

We next prove Euclidean volume growth. After a translation, we may assume $0 \in \Sigma$. For a.e. $R > 0$, the intersection

$$\Gamma_R := \Sigma \cap \partial B_R$$

is a smooth closed $(n-1)$-dimensional submanifold of the sphere $\partial B_R = S^n_R$. Since

$$H_{n-1}(S^n) = 0,$$

the cycle Γ_R bounds an n-dimensional region $D_R \subset S^n_R$. Choosing the smaller of the two sides of $S^n_R \setminus \Gamma_R$, we may assume

$$|D_R| \le \frac{1}{2}|S^n_R| \le C_nR^n.$$

Since Σ is area-minimizing and

$$\partial(\Sigma \cap B_R) = \Gamma_R = \partial D_R,$$

we obtain

$$|\Sigma \cap B_R| \le |D_R| \le C_nR^n$$

for a.e. $R > 0$.

Hence, by the Schoen–Simon–Yau theorem, Σ is a hyperplane. Since Σ is a graph over $\mathbb{R}^n$, that hyperplane cannot be vertical. Thus

$$u(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n + c$$

◆

2.2 Classification of Stable Minimal Cones for $n \le 6$

Proposition 2.2.1

Suppose $\mathbf{C}$ *is a minimal cone in* $\mathbb{R}^{n+1}$*, smooth away from the origin. Then we have the following Simons-type inequality on* $\mathbf{C} \setminus \{0\}$*:*

$$\frac{1}{2}\Delta|A|^2 \geq -|A|^4 + 2\frac{|A|^2}{r^2} + |\nabla|A||^2.$$

♠

Proof Choose $\{e_i\}$ to be a local orthonormal frame on $\mathbf{C} \setminus \{0\}$ such that e_n is the radial direction ∂_r. Recall the Simons identity for minimal hypersurfaces in $\mathbb{R}^{n+1}$:

$$\frac{1}{2}\Delta|A|^2 = -|A|^4 + |\nabla A|^2.$$

It suffices to show

$$|\nabla A|^2 \geq 2\frac{|A|^2}{r^2} + |\nabla|A||^2.$$

Since $\mathbf{C}$ is a cone, we have $A_{in} = 0$ and $A_{ij,n} = -\frac{1}{r}A_{ij}$ for $i, j < n$. Indeed,

$$A(rp) = \frac{A(p)}{r}.$$

Taking derivative in the radial direction gives

$$A_{ij,n}(rp) = -\frac{A_{ij}(p)}{r^2} = -\frac{1}{r}A_{ij}(rp).$$

So we have

$$\sum_{i,j,k} A_{ij,k}^2 = \sum_{\alpha,\beta=1}^{n-1} 3A_{\alpha\beta,n}^2 + \sum_{\alpha,\beta,\gamma=1}^{n-1} A_{\alpha\beta,\gamma}^2 = 2\frac{|A|^2}{r^2} + \sum_{\alpha,\beta=1}^{n-1} A_{\alpha\beta,n}^2 + \sum_{\alpha,\beta,\gamma=1}^{n-1} A_{\alpha\beta,\gamma}^2.$$

At a fixed point, choose $\{e_\alpha\}_{\alpha=1}^{n-1}$ so that

$$A_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}.$$

Then

$$|\nabla|A||^2 = \frac{1}{|A|^2}\sum_{k=1}^{n}\Big(\sum_{\alpha=1}^{n-1}\lambda_\alpha A_{\alpha\alpha,k}\Big)^2 = \frac{1}{|A|^2}\sum_{\beta=1}^{n-1}\Big(\sum_{\alpha=1}^{n-1}\lambda_\alpha A_{\alpha\alpha,\beta}\Big)^2 + \frac{1}{|A|^2}\Big(\sum_{\alpha=1}^{n-1}\lambda_\alpha A_{\alpha\alpha,n}\Big)^2.$$

By Cauchy–Schwarz and $A_{\alpha\alpha,n} = -\frac{1}{r}A_{\alpha\alpha}$,

$$|\nabla|A||^2 \leq \sum_{\alpha,\beta=1}^{n-1} A_{\alpha\alpha,\beta}^2 + \frac{|A|^2}{r^2} \leq \sum_{\alpha,\beta,\gamma=1}^{n-1} A_{\alpha\beta,\gamma}^2 + \sum_{\alpha,\beta=1}^{n-1} A_{\alpha\beta,n}^2.$$

Therefore,

$$|\nabla A|^2 = 2\frac{|A|^2}{r^2} + \sum_{\alpha,\beta=1}^{n-1} A_{\alpha\beta,n}^2 + \sum_{\alpha,\beta,\gamma=1}^{n-1} A_{\alpha\beta,\gamma}^2 \geq 2\frac{|A|^2}{r^2} + |\nabla|A||^2.$$

This completes the proof.

◆

Remark 2.2.2. A similar computation yields

$$\frac{1}{2}\Delta|A|^2 \geq p|\nabla|A||^2 + (3-p)\frac{|A|^2}{r^2} - |A|^4,$$

for any $p \leq 1 + \frac{2}{n-1}$.

Theorem 2.2.3 (Simons [Sim68])

Let $\mathbf{C}^n \subset \mathbb{R}^{n+1}$ *be a minimal nonflat cone, smooth away from the origin. If* $n \le 6$, $\mathbf{C}$ *is unstable. If* $n \ge 7$, *there exists a stable minimal nonflat cone.* ♡

Proof Suppose $\mathbf{C}$ is stable. Testing the stability inequality with $|A|\varphi$, we get

$$\begin{aligned}\int |A|^4\varphi^2 &\le \int |\nabla(|A|\varphi)|^2 = \int |\nabla|A||^2\varphi^2 + |A|^2|\nabla\varphi|^2 + \frac{1}{2}\langle \nabla|A|^2, \nabla\varphi^2\rangle \\ &\le \int |\nabla|A||^2\varphi^2 + |A|^2|\nabla\varphi|^2 - \frac{1}{2}\varphi^2\Delta|A|^2 \\ &\le \int |A|^2|\nabla\varphi|^2 + \varphi^2|A|^4 - 2\frac{|A|^2}{r^2}\varphi^2.\end{aligned}$$

Hence, we obtain

$$2\int \frac{|A|^2}{r^2}\varphi^2 \le \int |A|^2|\nabla\varphi|^2.$$

Now choose

$$\varphi = \max\{1, r\}^{1-\frac{n}{2}-2\varepsilon} r^{1+\varepsilon}.$$

We need to verify that this function is admissible. We compute

$$\begin{aligned}\int |A|^2|\nabla\varphi|^2 &= (1+\varepsilon)^2\int_{\{r<1\}} |A|^2 r^{2\varepsilon} + (2-\frac{n}{2}-\varepsilon)^2\int_{\{r\ge 1\}} |A|^2 r^{-n+2-\varepsilon} \\ &= (1+\varepsilon)^2\int_0^1 dr\int_\Sigma |A_\Sigma|^2 r^{2\varepsilon-2+n-1} d\Sigma \\ &\quad + (2-\frac{n}{2}-\varepsilon)^2\int_1^\infty dr\int_\Sigma |A_\Sigma|^2 r^{-n-2\varepsilon+n-1} d\Sigma \\ &= (\int_0^1 r^{2\varepsilon+n-3}dr + (2-\frac{n}{2}-\varepsilon)^2\int_1^\infty r^{-2\varepsilon-1}dr)\int_\Sigma |A_\Sigma|^2 d\Sigma < +\infty\end{aligned}$$

Here, $\Sigma = \mathbf{C} \cap S^n$ is the link of the cone, which is a smooth closed minimal hypersurface in S^n. Hence, φ is admissible. On the other hand, we have

$$2\int \frac{|A|^2}{r^2}\varphi^2 = 2\int_{\{r<1\}} |A|^2 r^{2\varepsilon} + 2\int_{\{r\ge 1\}} |A|^2 r^{-n+2-4\varepsilon}$$

When $n \le 6$, we can choose $\varepsilon > 0$ small enough such that $2 > (1+\varepsilon)^2$ and $2 > (2-\frac{n}{2}-\varepsilon)^2$. Hence $|A|^2 \equiv 0$, so $\mathbf{C}$ is flat, a contradiction. Therefore, $\mathbf{C}$ is unstable.

It remains to construct a stable minimal nonflat cone in $\mathbb{R}^8$.

Define the Simons-type cone by

$$\mathbf{C}_{p,q} = \left\{(x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : \, q|x|^2 = p|y|^2\right\}.$$

Lemma 2.2.4

$\mathbf{C}_{p,q}$ *is a minimal cone in* $\mathbb{R}^{p+q+2}$, *smooth away from the origin, and it is stable if and only if* $p+q \ge 6$. ♠

One can verify directly that $\mathbf{C}_{p,q}$ is minimal by computing its mean curvature. In particular, its principal curvatures are

$$\kappa_1 = \cdots = \kappa_p = \sqrt{\frac{q}{p}}, \qquad \kappa_{p+1} = \cdots = \kappa_{p+q} = -\sqrt{\frac{p}{q}}, \quad \text{and} \quad \kappa_{p+q+1} = 0.$$

Its second fundamental form satisfies $|A|^2 = \frac{n-1}{r^2}$. Choose $X = \frac{\varphi^2}{r^2}x$ and insert it into the first variation formula.

Then

$$0 = \int \operatorname{div}^{\mathbf{C}_{p,q}}(X) = \int n\frac{\varphi^2}{r^2} - 2\frac{\varphi^2}{r^4}|x|^2 + 2\frac{\varphi\nabla^{\mathbf{C}_{p,q}}\varphi\cdot x}{r^2} = \int (n-2)\frac{\varphi^2}{r^2} + 2\frac{\varphi\nabla^{\mathbf{C}_{p,q}}\varphi\cdot x}{r^2}.$$

Hence

$$\int (n-2)\frac{\varphi^2}{r^2} \le 2\sqrt{\int\frac{\varphi^2}{r^2}\int|\nabla^{\mathbf{C}_{p,q}}\varphi|^2} \implies \int\frac{(n-2)^2}{4}\frac{\varphi^2}{r^2} \le \int|\nabla^{\mathbf{C}_{p,q}}\varphi|^2.$$

Since $|A|^2 = \frac{n-1}{r^2}$, this can be rewritten as

$$\int\frac{(n-2)^2}{4(n-1)}|A|^2\varphi^2 \le \int|\nabla^{\mathbf{C}_{p,q}}\varphi|^2.$$

In particular, if $n \ge 7$, then $\frac{(n-2)^2}{4(n-1)} \ge 1$, so $\mathbf{C}_{p,q}$ is stable. ♦

In fact, one has the stronger result:

Theorem 2.2.5

Each $\mathbf{C}_{p,q}$ *is area-minimizing if* $p+q \ge 6$ *except for the case* $p, q = (1,5)$ *or* $(5,1)$. ♡

The case $\mathbf{C}_{p,p}$ with $p \ge 3$ was proved by Bombieri–De Giorgi–Giusti [BDGG69]. Lawson [Law72] later proved the result for $p+q > 6$, and Simões [Sim74] handled the remaining cases $p, q = (2,4)$ or $(4,2)$.

The original proof is quite involved and relies on calibrations. One seeks a vector field ξ such that $\operatorname{div}^{\mathbf{C}}\xi = 0$ and $\xi = \nu_{\mathbf{C}}$ on $\mathbf{C}\setminus\{0\}$, which implies that $\mathbf{C}$ is area-minimizing. Finding such a calibration for $\mathbf{C}_{p,q}$ is difficult and requires solving an ODE.

Here we present a different proof due to De Philippis–Paolini [DPP09], which uses a sub-calibration argument together with an explicit construction of a sub-calibration for $\mathbf{C}_{p,p}$.

2.3 Sub-minimal Sets and Sub-calibrations

Let $\Omega \subset \mathbb{R}^n$ be an open set. For a set E with smooth boundary, the perimeter $P(E, A)$ in a smooth bounded open set $A \subset \Omega$ is the $(n-1)$-dimensional Hausdorff measure of $\partial E \cap A$.

Definition 2.3.1

A set E with smooth boundary is sub-minimal *in Ω if for every smooth bounded open set $A \subset \Omega$ and every $F \subset E$ with $E \setminus F \subset\subset A$,*

$$P(E, A) \le P(F, A).$$

♣

Proposition 2.3.2

If E and $E^c = \Omega \setminus E$ are both sub-minimal in Ω, then E is minimal in Ω. ♠

Proof Let $A \subset \Omega$ be a smooth bounded open set, and let F satisfy $E \triangle F \subset\subset A$. Define $F' = E \cap F \subset E$ and $F'' = (E \cup F)^c \subset E^c$. Then $E \setminus F' \subset\subset A$ and $E^c \setminus F'' \subset\subset A$. By sub-minimality:

$$P(E, A) \le P(F', A), \quad P(E^c, A) \le P(F'', A).$$

Since $P(E^c, A) = P(E, A)$ and $P(F'', A) = P(E \cup F, A)$, we have

$$2P(E, A) \le P(E \cap F, A) + P(E \cup F, A).$$

Using the identity

$$P(E \cap F, A) + P(E \cup F, A) \le P(E, A) + P(F, A),$$

we conclude $P(E, A) \leq P(F, A)$, so E is minimal. ♦

Definition 2.3.3

Let E have smooth boundary. A C^1 vector field ξ on Ω is a sub-calibration *of E if:*

1. $\xi(x) = \nu_E(x)$ *(exterior unit normal) for all $x \in \partial E$;*
2. $\operatorname{div} \xi(x) \leq 0$ *for all $x \in E$;*
3. $|\xi(x)| \leq 1$ *for all $x \in \Omega$.* ♣

Theorem 2.3.4

If E admits a sub-calibration ξ, then E is sub-minimal. ♡

Proof Let A be a bounded open set, and let $F \subset E$ with $E \setminus F \subset\subset A$. Choose $\eta_j \in C_c^1(A)$ with $\eta_j = 1$ on $E \setminus F$, $0 \leq \eta_j \leq 1$, and $\bigcup_j \{x : \eta_j(x) = 1\} = A$. Let $\xi_j = \eta_j \xi$. Then

$$\int_{E\cap A} \operatorname{div} \xi_j - \int_{F\cap A} \operatorname{div} \xi_j = \int_{E\setminus F} \operatorname{div} \xi \leq 0,$$

so

$$\int_{E\cap A} \operatorname{div} \xi_j \leq \int_{F\cap A} \operatorname{div} \xi_j \leq P(F, A).$$

By the divergence theorem:

$$\int_{E\cap A} \operatorname{div} \xi_j = \int_{\partial E\cap A} \langle \xi_j, \nu_E \rangle d\mathcal{H}^{n-1} = \int_{\partial E\cap A} \eta_j d\mathcal{H}^{n-1} \geq \mathcal{H}^{n-1}\big(\partial E \cap \{\eta_j = 1\}\big).$$

Taking $j \to \infty$ yields $P(E, A) \leq P(F, A)$, so E is sub-minimal. ♦

Let $n = 2m$. The Simons cone is

$$\mathbf{C} = \big\{(x_1, \ldots, x_m, y_1, \ldots, y_m) \in \mathbb{R}^{2m} : x_1^2 + \cdots + x_m^2 = y_1^2 + \cdots + y_m^2\big\}.$$

Define

$$\mathcal{C} = \big\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m : |x| \leq |y|\big\}, \quad f(x, y) = \frac{1}{4}\big(|x|^4 - |y|^4\big).$$

Define the vector field

$$\xi = \frac{Df}{|Df|}.$$

Proposition 2.3.5

ξ is a sub-calibration for E in $\mathbb{R}^{2m} \setminus \{0\}$, where $E := \{(x, y) : |x| \leq |y|\}$, and $-\xi$ is a sub-calibration for E^c in $\mathbb{R}^{2m} \setminus \{0\}$. ♠

Proof $|\xi| = 1$ everywhere, and ξ is the exterior normal to E on ∂E. We compute

$$Df = (|x|^2 x, \, -|y|^2 y), \qquad |Df|^2 = |x|^6 + |y|^6.$$

Hence

$$\Delta f = (m + 2)|x|^2 - (m + 2)|y|^2, \quad D|Df|^2 = 6\big(|x|^4 x, \, |y|^4 y\big).$$

Thus

$$\begin{aligned}|Df|^3\mathrm{div}\xi &= |Df|^2\Delta f - \frac{1}{2}\langle D|Df|^2, Df\rangle \\ &= \big((m+2)|x|^2-(m+2)|y|^2\big)(|x|^6+|y|^6) - 3\big(|x|^8+|y|^8\big) \\ &= (m-1)|x|^8-(m-1)|y|^8+(m+2)|x|^2|y|^6-(m+2)|x|^6|y|^2 \\ &= (m-1)(|x|^4-|y|^4)(|x|^4+|y|^4)+(m+2)|x|^2|y|^2(|y|^4-|x|^4) \\ &= (|x|^4-|y|^4)\big((m-1)(|x|^4+|y|^4)-(m+2)|x|^2|y|^2\big).\end{aligned}$$

Note that

$$(m-1)a^2-(m+2)ab+(m-1)b^2$$

is nonnegative if and only if

$$\det\begin{pmatrix} m-1 & -\frac{m+2}{2} \\ -\frac{m+2}{2} & m-1\end{pmatrix} = (m-1)^2-\frac{(m+2)^2}{4} = \frac{3m(m-4)}{4} \geq 0.$$

◆

Theorem 2.3.6

The Simons cone $\mathbf{C}$ *is area-minimizing in* $\mathbb{R}^8$.

♡

Proof By the preceding theorem, E and E^c are sub-minimal in $\mathbb{R}^8 \setminus \{0\}$. Since the origin has codimension > 1, the perimeter is unchanged, and hence E and E^c are sub-minimal in $\mathbb{R}^8$. By the preceding proposition, E is minimal, so its boundary $\mathbf{C}$ is area-minimizing.

◆

Remark 2.3.7. One may consider the gradient of the function

$$\frac{1}{4}\left(q^2|x|^4-p^2|y|^4\right)$$

in the case of the minimizing cone $\mathbf{C}_{p,q}$.

Chapter 3 Background on Geometric Measure Theory

We collect some basic definitions and facts about the geometric measure theory. Readers can refer to [All72, Sim83] for more details.

3.1 Varifolds

Definition 3.1.1 (Varifold)

*An n-**varifold** V in $\mathbb{R}^{n+k}$ is a Radon measure on $\mathbb{R}^{n+k} \times G(n+k,n)$, where $G(n+k,n)$ is the set of all n-dimensional subspaces in $\mathbb{R}^{n+k}$.*

♣

Example 3.1 (Varifold associated with a submanifold). Let $M \subset \mathbb{R}^{n+k}$ be an n-dimensional manifold, and let θ be an $\mathcal{H}^n$-measurable function on M. Then the n-varifold $V = |(M,\theta)|$ is defined by

$$V(U) = \int_{(x,T_xM)\in U} \theta(x)d\mathcal{H}^n|_M(x).$$

Equivalently, for any continuous compactly supported function f, we have

$$V(f) = \int \theta(x)f(x,T_xM)d\mathcal{H}^n|_M(x).$$

Thus, a varifold arising from an n-dimensional manifold records tangent-plane information as well. If $\theta = 1$, we usually write $|(M,1)| = |M|$. For example, P is the tangent plane of M at x if and only if the measure $|M|$ restricted to $\{x\} \times G(n+k,n)$ is nonzero.

Definition 3.1.2 (Weight measure, support, and density)

*The **weight measure** $\|V\|$ of an n-varifold V is defined by*

$$\|V\|(A) = V(A \times G(n+m,n)),$$

for any Borel subset $A \subset \mathbb{R}^{n+m}$. Hence, $\|V\|$ is a Radon measure on $\mathbb{R}^{n+m}$. The support of $\|V\|$, denoted by $\operatorname{spt}\|V\|$, is defined by

$$\operatorname{spt}\|V\| = \left\{x \in \mathbb{R}^{n+k} : \|V\|(B_r^{n+k}(x)) > 0 \text{ for any } r > 0\right\}.$$

The (n-dimensional) density of $\|V\|$ at x is defined by

$$\Theta(\|V\|,x) = \lim_{r\to 0+} \frac{\|V\|(B_r^{n+k}(x))}{\omega_n r^n},$$

if the limit exists, where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

♣

Theorem 3.1.3 (Compactness)

Suppose V_i is a sequence of n-varifolds such that for every compact $K \subset \mathbb{R}^{n+k}$, there exists $C = C(K)$ with

$$V_i(K \times G(n+k,n)) \le C,$$

then, up to a subsequence, we can find $V_i \to V$ in the sense of Radon measures. (The convergence is in the varifold sense.) Equivalently,

$$\lim_{i\to\infty} V_i(f) = V(f)$$

for any $f \in C_c(\mathbb{R}^{n+k} \times G(n+k,n))$. ♡

Example 3.2 (Diffuse limit of one-dimensional varifolds). Suppose V_n is defined by

$$\sum_{i=1}^{2^n} |([0,1] \times \{\frac{i}{2^n}\}, \frac{i}{2^n})|,$$

Then it converges to V in the varifold sense, where

$$V(f) = \int_0^1 \int_0^1 f(x, y, \{x_2 = 0\}) dx dy.$$

Note that $\operatorname{spt}\|V\| = [0,1]^2$, so V cannot be written as $V = |(M, \theta)|$ for any one-dimensional manifold M.

Definition 3.1.4 (Countable rectifiability)

We say M *is* ***countably*** n***-rectifiable*** *if* $M \subset N \cup \bigcup_{j=1}^{\infty} N_j$ *where* $\mathcal{H}^n(N) = 0$ *and each* N_j *is an* n*-dimensional embedded* C^1 *submanifold of* $\mathbb{R}^{n+k}$.

Equivalently, M *is countably* n*-rectifiable if and only if there exists a countable family of Lipschitz maps* $f_j : \mathbb{R}^n \to \mathbb{R}^{n+k}$ *such that*

$$M = N \cup \bigcup_{j=1}^{\infty} f_j(A_j),$$

where $A_j \subset \mathbb{R}^n$ *and* $\mathcal{H}^n(N) = 0$. ♣

For any countably n-rectifiable set M, we write $T_x M$ for the approximate tangent space of M.

Definition 3.1.5 (Approximate tangent space)

Let M *be an* $\mathcal{H}^n$*-measurable subset of* $\mathbb{R}^{n+k}$ *with* $\mathcal{H}^n(M \cap K) < +\infty$ *for every compact subset* K*. We say that an* n*-dimensional subspace* P *is an* ***approximate tangent space*** *of* M *at* x *if and only if*

$$\lim_{r \to 0+} \int_{\eta_{x,r}(M)} f(y) d\mathcal{H}^n(y) = \int_P f(y) d\mathcal{H}^n(y), \quad \textit{for any } f \in C_c(\mathbb{R}^{n+k}).$$

♣

Theorem 3.1.6

If M *is rectifiable, then for* $\mathcal{H}^n$*-almost every* $x \in M$*, there exists a unique approximate tangent space* $T_x M$ *of* M *at* x. ♡

Definition 3.1.7 (Rectifiable varifold)

We say an n*-varifold* V *is* ***rectifiable*** *if there exists a countably* n*-rectifiable,* $\mathcal{H}^n$*-measurable subset* M *of* $\mathbb{R}^{n+k}$ *and a positive locally* $\mathcal{H}^n$*-integrable function* θ *on* M *such that*

$$V(f) = \int_M f(x, T_x M) \theta(x) d\mathcal{H}^n(x)$$

We use the notation $V = |(M, \theta)|$ *for the varifold associated with* M *and* θ. ♣

Definition 3.1.8 (Pushforward of a varifold)

Let V *be an* n*-varifold in* $\mathbb{R}^{n+k}$ *and let* $F : \mathbb{R}^{n+k} \to \mathbb{R}^{m+l}$ *be a* C^1 *map. The pushforward varifold* $F_\# V$ *is defined by*

$$F_\# V(\phi) = \int \phi(F(x), DF_x(S)) \, |J_F(x, S)| \, dV(x, S),$$

for any continuous function ϕ with compact support on $\mathbb{R}^{m+l} \times G(m+l,n)$, where $J_F(x,S)$ is the Jacobian of F restricted to S, i.e.,

$$J_F(x,S) = \sqrt{\det\left(\frac{\partial F}{\partial \tau_i} \cdot \frac{\partial F}{\partial \tau_j}\right)},$$

where $\{\tau_i\}_{i=1}^n$ is an orthonormal basis of S. ♣

Let F_t be a one-parameter family of diffeomorphisms on $\mathbb{R}^{n+k}$ with F_0 being the identity map. The first variation of an n-varifold V under the variation F_t is defined by

$$\left.\frac{d}{dt}\right|_{t=0} \|(F_t)_\# V\|(K),$$

for any compact subset $K \subset \mathbb{R}^{n+k}$.

Theorem 3.1.9 (First Variation Formula)

We have

$$\left.\frac{d}{dt}\right|_{t=0} \|(F_t)_\# V\|(K) = \int \operatorname{div}^S \varphi(x) dV(x,S).$$

♡

Definition 3.1.10 (First variation of a varifold)

We define the ***first variation*** *of V as a linear functional δV on $C_c(\mathbb{R}^{n+k}, \mathbb{R}^{n+k})$ by*

$$\delta V(\varphi) = \int \operatorname{div}^S \varphi(x) dV(x,S).$$

♣

Definition 3.1.11 (Bounded first variation)

We say that V has ***bounded first variation*** *if δV is a bounded linear functional on $C_c(\mathbb{R}^{n+k}, \mathbb{R}^{n+k})$. Hence, by the Riesz representation theorem, there exists a Radon measure $\|\delta V\|$ and a $\|\delta V\|$-measurable vector-valued function ν_V such that $|\nu_V(x)| = 1$ for $\|\delta V\|$-almost every x and*

$$\delta V(\varphi) = \int \langle \nu_V(x), \varphi(x) \rangle d\|\delta V\|(x).$$

In particular, we can decompose $\|\delta V\| = h\|V\| + \sigma_V$ into the absolutely continuous part $h\|V\|$ and the singular part σ_V with respect to $\|V\|$. ♣

Remark 3.1.12. Let μ and ν be two Radon measures on $\mathbb{R}^{n+k}$. We say μ is *absolutely continuous* with respect to ν (denoted $\mu \ll \nu$) if for every Borel set A, $\nu(A) = 0$ implies $\mu(A) = 0$. By the Radon-Nikodym theorem, if $\mu \ll \nu$, then there exists a ν-measurable function h such that

$$\mu(A) = \int_A h \, d\nu$$

for all Borel sets A.

Conversely, a measure μ is *singular* with respect to ν if there exists a Borel set A such that $\nu(A) = 0$ and $\mu(\mathbb{R}^{n+k} \setminus A) = 0$. Intuitively, μ and ν are supported on disjoint sets.

Any Radon measure μ can be uniquely decomposed as

$$\mu = \mu_{\text{ac}} + \mu_{\text{sing}},$$

where $\mu_{\text{ac}} \ll \nu$ and $\mu_{\text{sing}} \perp \nu$.

So, if V has bounded first variation, we can write

$$\delta V(\varphi) = -\int \langle \vec{H}, \varphi(x)\rangle d\|V\|(x) + \int \langle \nu_V(x), \varphi(x)\rangle d\sigma_V(x).$$

where $\vec{H} = -h\nu_V$ is called the generalized mean curvature vector of V, and $\nu_V\sigma_V$ is called the generalized boundary of V.

Example 3.3. Suppose $M = \{(\cos\theta, \sin\theta) : \theta \in (0,\pi)\}$, the 1-dimensional half-circle in $\mathbb{R}^2$, and $V = |M|$ is the associated varifold. Then, we have

$$\begin{aligned}\delta V(\varphi) = \int_M \operatorname{div}^M \varphi(x) d\mathcal{H}^1|_M(x) &= \int_M \operatorname{div}^M \varphi^\top(x) d\mathcal{H}^1|_M(x) + \int_M \varphi(x)\cdot x \\ &= \varphi(1,0)\cdot(0,-1) + \varphi(-1,0)\cdot(0,-1) + \int_M \varphi(x)\cdot x d\mathcal{H}^1|_M(x).\end{aligned}$$

So the generalized mean curvature vector $\vec{H} = -x$ is just the usual mean curvature vector of M, and the generalized boundary $\nu_V\sigma_V = (0,-1)\delta_{(1,0)} + (0,-1)\delta_{(-1,0)}$ represents the two boundary points of M with the corresponding outward normal vectors.

Example 3.4. Suppose $M = [0,1]\times\{0\}, \theta(x,y) = x$, and $V = |(M,\theta)|$ is the associated varifold. Then, we have

$$\delta V = \int_0^1 \frac{\partial}{\partial x}\varphi^\top(x,0) x dx = \varphi^\top(1,0)\cdot(1,0) - \int_0^1 \varphi^\top(x,0)\cdot(1,0) dx.$$

So we have

$$\vec{H} = (\frac{1}{x}, 0), \quad \nu_V\sigma_V = (1,0)\delta_{(1,0)}.$$

Thus, the generalized mean curvature vector $\vec{H}$ depends not only on the geometry of M, but also on the weight function θ.

This example also shows that, unlike the case of smooth submanifolds, the generalized mean curvature vector of a varifold may not be perpendicular to the tangent plane. If we restrict to integral rectifiable varifolds, we have the following result of Brakke [Bra15].

Theorem 3.1.13

Suppose V is an integral rectifiable varifold with bounded first variation, then the generalized mean curvature vector $\vec{H}$ is perpendicular to the tangent plane of V for $\|V\|$-almost every point.

Definition 3.1.14 (Stationary varifold)

We say that V is a ***stationary varifold*** *in U if for any $\varphi \in C_c(U, \mathbb{R}^{n+k})$, we have*

$$\int \operatorname{div}^S \varphi(x) dV(x,S) = 0.$$

If $V = |M|$, then V being stationary means

$$\int_M \operatorname{div}^M \varphi(x) d\mathcal{H}^n|_M(x) = 0.$$

This is equivalent to saying that M is minimal.

This notion of stationarity is very weak. Triple-junctions of three half-planes meeting at 120 degrees are stationary.

Theorem 3.1.15 (Monotonicity Formula for Stationary Varifolds)

Let V be a stationary n-varifold in $\mathbb{R}^{n+k}$. Then the function

$$\Phi(r) = \frac{\|V\|(B_r(x_0))}{\omega_n r^n}$$

is monotone increasing in $r > 0$. Moreover, we have

$$\Phi(r) - \Phi(s) = \int_{B_r(x_0)\backslash B_s(x_0)} \frac{|(y-x_0)^{\perp}|^2}{|y-x_0|^{n+2}} d\|V\|(y)$$

♡

Proof We can choose test vector fields φ as before. Suppose $x_0 = 0$ for simplicity.

$$\varphi = \begin{cases} y\left(\frac{1}{|y|^n} - \frac{1}{\rho^n}\right), & \sigma \leq |y| \leq \rho \\ y\left(\frac{1}{\sigma^n} - \frac{1}{\rho^n}\right), & |y| < \sigma \end{cases}$$

We insert φ into the first variation formula and get the desired monotonicity formula. ◆

Remark 3.1.16. One can obtain a modified monotonicity formula for varifolds with L^p-integrable mean curvature ($p > n$) and no generalized boundary.

Theorem 3.1.17 (Compactness of stationary varifolds, [All72, Sim83])

Suppose V_i is a sequence of rectifiable stationary varifolds in U and for any $K \subset\subset U$, $\sup \|V_i\|(K) < +\infty$. We also assume $\Theta(\|V_i\|, x) \geq 1$ for almost every $x \in \operatorname{spt}\|V_i\|$. Then, up to a subsequence, there exists a rectifiable stationary varifold V in U such that $V_i \to V$ in the varifold sense.

In particular, if each V_i is integral, so is V.

♡

- The condition $\Theta \geq 1$ is essential.
- Stationarity is also essential: $V_n = \sum_{i=1}^{2^n} |[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}]|$ converges to $|([0,1], \frac{1}{2})|$.

Definition 3.1.18 (Tangent cones of a varifold)

*The **tangent cone** of an n-varifold V at x, denoted by $\operatorname{VarTan}(V, x)$, is defined by*

$$\operatorname{VarTan}(V, x) := \{V' : V' = \lim_{i\to\infty} (\eta_{x,\rho_i})_{\#} V \text{ for some } \rho_i \to 0^+\}.$$

*The **tangent cone at infinity**, denoted by $\operatorname{VarTan}(V, \infty)$, is defined by*

$$\operatorname{VarTan}(V, \infty) := \{V' : V' = \lim_{i\to\infty} (\eta_{0,\rho_i})_{\#} V \text{ for some } \rho_i \to +\infty\}.$$

♣

Remark 3.1.19. The tangent cone of a varifold may not be unique. Right now, we do not even know if the tangent cone of any stationary rectifiable varifold is unique or not. This is still an open problem.

Proposition 3.1.20

If V is stationary in $\mathbb{R}^{n+k}$, then the tangent cone of V at ∞ or x is a stationary cone.

♠

Finally, for any n-varifold defined on U, $\operatorname{reg}\|V\|$ denotes the regular set of $\operatorname{spt}\|V\|$ in U, i.e., the set of points $x \in \operatorname{spt}\|V\|$ such that there exists $r > 0$ with $B_r^{n+1}(x) \subset U$ and $\operatorname{spt}\|V\| \cap B_r^{n+1}(x)$ is a smooth (immersed) hypersurface in $B_r^{n+1}(x)$. $\operatorname{sing}\|V\|$ denotes the singular set of $\operatorname{spt}\|V\|$ in U.

Allard's regularity theorem is a foundational result in geometric measure theory. It gives conditions under which a stationary varifold is regular (i.e., smooth) near a point.

Theorem 3.1.21 (Allard's Regularity Theorem, [All72, Sim83])

Let V *be an* n*-dimensional stationary integral rectifiable varifold in* $B_2(0) \subset \mathbb{R}^{n+k}$*, and suppose that*

$$\|V\|(B_2(0)) \leq (1+\delta)\omega_n 2^n$$

for some $0 < \delta < 1$*. Then, there exists* $\varepsilon = \varepsilon(n, \delta)$ *such that if*

$$E^2 := \int_{B_2(0)} \operatorname{dist}^2(x, P)\, d\|V\|(x) \leq \varepsilon$$

where $P = \{x_{n+1} = 0\}$*, then there exists a function* $u \in C^{1,\alpha}(B_1^n(0))$ *such that*

$$\operatorname{spt}\|V\| \cap B_1(0) = \{(x', u(x')) : x' \in B_1^n(0)\}$$

and

$$\sup_{B_1^n(0)} |u| + \|Du\|_{L^\infty(B_1^n(0))} + [Du]_{C^{0,\alpha}(B_1^n(0))} \leq C(n, \delta)E.$$

♡

Note that δ cannot be 1 since otherwise, we have the scaled catenoid as a counterexample.

Theorem 3.1.22 (Density-one regularity, [All72, Sim83])

Let V *be an* n*-dimensional stationary integral rectifiable varifold in an open set* $U \subset \mathbb{R}^{n+k}$*. Suppose that the density of* $\|V\|$ *at a point* $x_0 \in U$ *is* 1*. Then* $x_0 \in \operatorname{reg}\|V\|$*.*

♡

3.2 Sets of Finite Perimeter

We collect some basic definitions and facts about the set of finite perimeter.

If E is a set with smooth boundary, then we have the following Gauss-Green formula:

$$\int_E \operatorname{div}\varphi d\mathcal{H}^n = \int_{\partial E} \varphi \cdot \nu d\mathcal{H}^{n-1},$$

for any $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$. In particular, if we require that $|\varphi| \leq 1$, then the right-hand side can be bounded by the perimeter of E as

$$\int_{\partial E} \varphi \cdot \nu d\mathcal{H}^{n-1} \leq \mathcal{H}^{n-1}(\partial E).$$

and equality holds if and only if $\varphi = \nu$ on ∂E. This fact motivates the following definition of the perimeter of E as

$$|\partial E| = \sup_{\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n), |\varphi| \leq 1} \int_{\partial E} \varphi \cdot \nu d\mathcal{H}^{n-1}.$$

Note that the right-hand side of the equation above does not depend on the regularity of the boundary of E. This motivates the following definition.

Definition 3.2.1 (Perimeter and locally finite perimeter)

The ***perimeter*** *of a set* E *in an open set* U *is defined as*

$$P(E, U) = \sup_{\varphi \in C_c^1(U, \mathbb{R}^n), |\varphi| \leq 1} \int_{U \cap E} \operatorname{div}\varphi d\mathcal{H}^n.$$

We say that E *has* ***locally finite perimeter*** *in* U *if* $P(E, W) < +\infty$ *for any* $W \subset\subset U$*. Such a set is also called a* ***Caccioppoli set****.*

♣

Suppose E has locally finite perimeter in U. Then, we can consider the linear functional J_E on $C_c^1(U, \mathbb{R}^n)$

defined by

$$J_E(\varphi) = \int_{U\cap E} \mathrm{div}\varphi d\mathcal{H}^n.$$

This functional is clearly linear. Since we have assumed that E has locally finite perimeter, we have that J_E is a bounded linear functional on $C_c^1(W, \mathbb{R}^n)$ for any $W \subset\subset U$. (Recall that an operator T is bounded if $\sup_{|\varphi|\leq 1} |T(\varphi)| < +\infty$.) Now, we can apply the Riesz Representation Theorem to get a unique Radon measure μ_E on U, and a vector-valued function ν_E on U such that $|\nu_E(x)| = 1$ for μ_E-almost every x and

$$J_E(\varphi) = \int_U \varphi \cdot \nu_E d\mu_E,$$

for any $\varphi \in C_c^1(U, \mathbb{R}^n)$.

The vector-valued measure $\nu_E\mu_E$ is called the Gauss-Green measure of E in U. We use $\overrightarrow{\mu}_E$ to denote the vector-valued measure $\nu_E\mu_E$. Then, the perimeter of E in U can also be written as

$$P(E, U) = \mu_E(U).$$

We can understand μ_E as a boundary measure of E in U, and ν_E as a boundary normal vector of E in U, pointing outward.

Example 3.5 (A quadrant). Suppose $E = [0, +\infty)^2 \in \mathbb{R}^2$. Then, E is a set with locally finite perimeter. In particular,

$$\mu_E = \mathcal{H}^1|_{[0,+\infty)\times\{0\}} + \mathcal{H}^1|_{\{0\}\times[0,+\infty)}, \quad \nu_E = (0, -1)|_{[0,+\infty)\times\{0\}} + (1, 0)|_{\{0\}\times[0,+\infty)}.$$

Recall that $\mathrm{spt}\,\mu_E$ is the support of μ_E, which is defined as the set of points x such that $\mu_E(B_r(x)) > 0$ for any $r > 0$.

Then, we have the following proposition:

Proposition 3.2.2

Suppose E is a set of locally finite perimeter in $\mathbb{R}^n$. Then, $\mathrm{spt}\,\mu_E \subset \partial E$, where ∂E is the topological boundary of E.

Remark 3.2.3 (Why $\mathrm{spt}\,\mu_E$ can be strictly smaller than ∂E). Let

$$E := (0, 1)^2 \backslash \{0.5\} \times [0, 0.5] \subset \mathbb{R}^2.$$

Then ∂E contains both the outer boundary of the square and the slit:

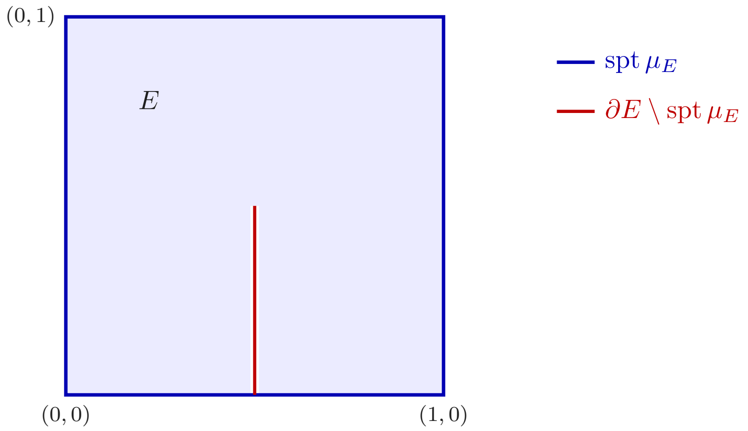
$$\partial E = \partial(0, 1)^2 \ \cup \ \big(\{0.5\} \times [0, 0.5]\big).$$

On the other hand, removing a 1-dimensional set does not change χ_E in L^1, so the perimeter measure is the same as for the open square:

$$\mu_E = \mu_{(0,1)^2}, \qquad \mathrm{spt}\,\mu_E = \partial(0, 1)^2.$$

Hence

$$\mathrm{spt}\,\mu_E \subsetneq \partial E.$$



To better illustrate the ”true” boundary of a set of finite perimeter in the measure sense, we introduce the reduced boundary.

Definition 3.2.4 (Reduced boundary)

Given a set E of locally finite perimeter in $\mathbb{R}^n$, the ***reduced boundary*** *$\partial^* E$ is the set of points $x \in \mathbb{R}^n$ such that the limit*

$$\nu_E(x) := \lim_{r \to 0^+} \frac{\vec{\mu}_E(B_r(x))}{\mu_E(B_r(x))}$$

exists and belongs to $\mathbb{S}^{n-1}$. ♣

For the $E = (0,1)^2 \backslash \{0.5\} \times [0,0.5]$, we have

$$\partial^* E = \partial(0,1)^2 \backslash \{(0,0),(1,0),(0,1),(1,1)\}.$$

In particular, we have

$$\lim_{r \to 0^+} \frac{\vec{\mu}_E(B_r(0))}{\mu_E(B_r(0))} = (-\frac{1}{2}, -\frac{1}{2}) \notin \mathbb{S}^{n-1}.$$

So $(0,0) \notin \partial^* E$.

We have the following structure theorem for the reduced boundary.

Theorem 3.2.5 (De Giorgi’s Structure Theorem)

Suppose E is a set of locally finite perimeter in $\mathbb{R}^n$. Then, the reduced boundary $\partial^* E$ is a $(n-1)$-countably rectifiable set in $\mathbb{R}^n$, and we actually have

$$\mu_E = \mathcal{H}^{n-1}|_{\partial^* E}, \quad \nu_E = \text{ the outer normal vector field of } \partial^* E.$$ ♡

Remark 3.2.6. Note that since $\partial^* E$ is rectifiable, so the approximate tangent space $T_x \partial^* E$ is well-defined for μ_E-almost every $x \in \partial^* E$. Hence, we can choose the outer normal vector field perpendicular to $T_x \partial^* E$ and point outward of E.

Definition 3.2.7 (Local convergence of sets)

Given Lebesgue measurable sets $\{E_h\}_{h \in \mathbb{N}}$ and E in $\mathbb{R}^n$, we say that E_h ***locally converges*** *to E, and write $E_h \xrightarrow{\text{loc}} E$, if*

$$\lim_{h \to \infty} |K \cap (E \Delta E_h)| = 0, \qquad \forall K \subset \mathbb{R}^n \text{ compact.}$$

This is equivalent to say that $\chi_{E_h} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$. ♣

Proposition 3.2.8 (Lower semicontinuity of perimeter)

If $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of sets of locally finite perimeter in $\mathbb{R}^n$, with

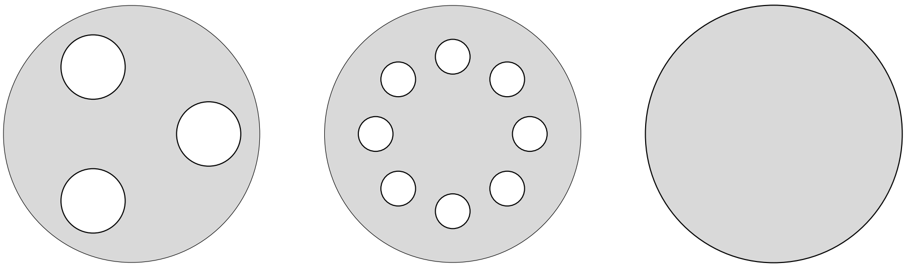
$$E_h \xrightarrow{\text{loc}} E, \qquad \limsup_{h\to\infty} P(E_h;K) < \infty,$$

for every compact set K in $\mathbb{R}^n$, then E is of locally finite perimeter in $\mathbb{R}^n$, $\mu_{E_h} \rightharpoonup^* \mu_E$ and, for every open set $A \subset \mathbb{R}^n$, we have

$$P(E;A) \le \liminf_{h\to\infty} P(E_h;A).$$

♠

Example 3.6 (Disappearing holes). The inequality above can be strict.



Let B denote the closed unit disc $B = \{x \in \mathbb{R}^2 : |x| \le 1\}$. For each $i \in \mathbb{N}$, let $E_i = B \setminus \bigcup_{k=1}^{n_i} \bar{B}_{r_i}(x_{i,k})$, where $\{x_{i,k}\}_{k=1}^{n_i}$ is a collection of centers inside B, n_i increases as $i \to \infty$, and $r_i \to 0$ in such a way that $n_i r_i \to c > 0$ and

$$\bigcup_{k=1}^{n_i} \bar{B}_{r_i}(x_{i,k}) \subset B,$$

with all the small discs disjoint and contained in B.

For each E_i, the perimeter is

$$P(E_i) = P(B) + n_i P(\bar{B}_{r_i}) = 2\pi + n_i \cdot 2\pi r_i,$$

since each removed disc adds $2\pi r_i$ to the perimeter.

As $i \to \infty$, the number of holes $n_i \to \infty$ and the radii $r_i \to 0$, so the total removed area goes to 0. In the limit, the set E_i converges (locally in measure, or in L^1_{loc}) to B. However,

$$\lim_{i\to\infty} P(E_i) = 2\pi + 2\pi c,$$

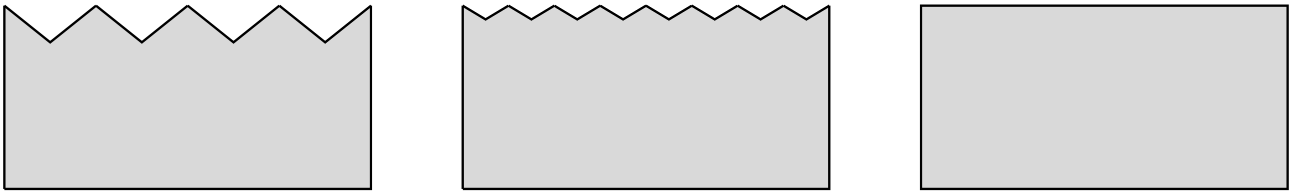
because the total boundary length of the small holes remains visible before passing to the limit.

Thus, the perimeter functional is *lower semicontinuous* under local convergence, and it is possible that

$$P(\lim_{i\to\infty} E_i) < \liminf_{i\to\infty} P(E_i),$$

because the limiting set does not retain the interior boundaries present in the approximating sequence.

Example 3.7 (Oscillating boundary).



Let $D = [0,4] \times [0,2] \subset \mathbb{R}^2$. For each $i \in \mathbb{N}$, construct E_i as the subset of D with lower and side boundaries of D, but the top replaced with a zig-zag curve of n_i "teeth," each of amplitude h_i (height above $y = 2$), so that as $i \to \infty$, $n_i \to \infty$ and $h_i \to 0$.

Each E_i is the set (shaded in the pictures above) contained below its zig-zag boundary and above $y = 0$. As $i \to \infty$, the upper boundary of E_i converges to the straight line $y = 2$, and E_i converges (in measure) to D.

The perimeter of E_i is

$$P(E_i) = 2 \cdot 2 + 4 + L_{\text{zigzag}},$$

where L_{zigzag} is the total length of the zig-zag curve. If each tooth projects horizontally $\delta x = 4/n_i$ and has vertical height h_i, then

$$L_{\text{zigzag}} = n_i \cdot 2\sqrt{\left(\frac{\delta x}{2}\right)^2 + h_i^2} = n_i \cdot 2\sqrt{(2/n_i)^2 + h_i^2}.$$

If $n_i h_i \to a > 0$, then the zig-zag boundary length satisfies

$$\lim_{i\to\infty} L_{\text{zigzag}} = 2\sqrt{4 + a^2} > 4.$$

Thus, in the limit, the region E_i converges to D, but the perimeter keeps a positive excess:

$$P(\lim_{i\to\infty} E_i) < \liminf_{i\to\infty} P(E_i).$$

This example again shows lower semicontinuity: the limiting domain loses the additional oscillating boundary length in the limit.

Theorem 3.2.9 (Compactness Theorem)

If $R > 0$ and $(E_k)_{k\in\mathbb{N}}$ are sets of finite perimeter in $\mathbb{R}^n$, with

$$\sup_{k\in\mathbb{N}} P(E_k) < \infty,$$

$$E_k \subset B_R, \qquad \forall\, k \in \mathbb{N},$$

then one can find a set E of finite perimeter in $\mathbb{R}^n$ and a subsequence of $(E_k)_{k\in\mathbb{N}}$, still denoted by E_k, such that

$$\chi_{E_k} \to \chi_E, \qquad \mu_{E_k} \overset{*}{\rightharpoonup} \mu_E, \qquad E \subset B_R.$$

♡

Definition 3.2.10 (Perimeter minimizer)

Suppose $A \subset \mathbb{R}^n$ is a bounded set and E_0 is a set of finite perimeter in $\mathbb{R}^n$. We say E_0 is a perimeter minimizer in A if

$$P(E_0; A) \le P(E; A)$$

for every set of finite perimeter E such that $E \setminus A = E_0 \setminus A$.

In particular, we say E_0 is a perimeter minimizer in $\mathbb{R}^n$ if E_0 is a perimeter minimizer for any bounded set A.

♣

Proposition 3.2.11 (Compactness of minimizer)

Suppose $\{E_k\}$ is a sequence of perimeter minimizers in $\mathbb{R}^n$, and assume E_k converges to E in $L^1_{\text{loc}}(\mathbb{R}^n)$. Then E is also a perimeter minimizer in $\mathbb{R}^n$.

♠

Now, we are ready to state the general existence of minimizers in the following sense.

Proposition 3.2.12 (Existence of minimizers for the Plateau-type problem)

Let $A \subset \mathbb{R}^n$ be a bounded set and let E_0 be a set of finite perimeter in $\mathbb{R}^n$. Then there exists a set of finite perimeter E such that $E \setminus A = E_0 \setminus A$ and

$$P(E) \leq P(F)$$

for every F such that $F \setminus A = E_0 \setminus A$.

Remark 3.2.13. For example, given a boundary curve Γ, which lies on a boundary of a convex domain D. We can construct a set of finite perimeter E such that $\partial(\partial E \cap \Omega) = \Gamma$ and it is the one with the minimal perimeter.

The key here is actually the regularity of the boundary of the set of finite perimeter.

3.3 Dimension Reduction and Regularity of Perimeter Minimizers

Theorem 3.3.1

Suppose E is a perimeter minimizer in $\mathbb{R}^{n+1}$. We denote $V = |\partial^* E|$. Then, we have the following regularity result: $\operatorname{sing}\|V\| = \emptyset$ if $n \leq 6$, $\operatorname{sing}\|V\|$ is discrete if $n = 7$, and $\mathcal{H}^{n-7+\delta}(\operatorname{sing}\|V\|) = 0$ for any $\delta > 0$ for $n \geq 8$.

Theorem 3.3.2

Let $\mathcal{V}$ be the collection of all the varifolds in the preceding theorem, i.e., the varifolds corresponding to the perimeter minimizers. Then,

$$\dim(\operatorname{sing}(\|V\| \cap B_1)) \leq n - 7.$$

In particular, if $n = 7$, then $\operatorname{sing}(\|V\| \cap B_1)$ is discrete.

Proof We denote $F^l = \{V \in \mathcal{V} : \mathcal{H}^l(\operatorname{sing} \cap B_1) > 0\}$.

Proposition 3.3.3

For each $V \in F^l$, there exists $\boldsymbol{C} \in \operatorname{VarTan}(V, x) \cap F^l$ for $\mathcal{H}^l-$a.e. $x \in \operatorname{sing}(\|V\|) \cap B_1$.

Proof Recall that we actually have for $\mathcal{H}^l$-a.e. $x \in \operatorname{sing}(\|V\|) \cap B_1$, we have

$$\limsup_{r \to 0} \frac{\mathcal{H}^l_\infty(\operatorname{sing}\|V\| \cap B_r(x))}{\omega_n r^l} > 0.$$

This is equivalent to $\mathcal{H}^l(A) > 0 \equiv \mathcal{H}^l_\infty(A) > 0$ for any $A \subset \mathbb{R}^n$.

We choose $r_i \to 0$ such that

$$\lim_{i \to \infty} \frac{\mathcal{H}^l_\infty(\operatorname{sing}\|V\| \cap B_{r_i}(x))}{\omega_n r_i^l} > 0.$$

By taking a subsequence, we can assume $(\eta_{x,r_i})_\# V$ converges to $\boldsymbol{C} \in \operatorname{VarTan}(V, x)$.

If $\mathcal{H}^l_\infty(\operatorname{sing}\|\boldsymbol{C}\|) = 0$, then for any $\varepsilon > 0$, we can find a covering of $\operatorname{sing}\|\boldsymbol{C}\|$ by balls $\{B_{s_j}(y_j)\}_{j=1}^\infty$ such that

$$\sum_{j=1}^\infty s_j^l < \varepsilon.$$

Note that $\operatorname{sing}\|\boldsymbol{C}\| \cap B_1(0)$ is compact, we know $\operatorname{sing}\|(\eta_{x,r_i})_\# V\| \cap B_1(0)$ can also be covered by $\{B_{s_j}(y_j)\}_{j=1}^\infty$

for i large enough. Thus, we have

$$\frac{\mathcal{H}^l_\infty(\operatorname{sing}\|V\|\cap B_{r_i}(x))}{r_i^l}<\varepsilon$$

for i large enough, which is a contradiction. ◆

Now we apply the above proposition iteratively to obtain a sequence of varifolds $\{V_k\}_{k=0}^K$ such that:

1. $V_0=V$.
2. $V_{k+1}\in \operatorname{VarTan}(V_k,x_k)$ for some $x_k\in \operatorname{sing}(\|V_k\|)\cap B_1\backslash\mathcal{S}(V_k)$.
3. $\dim(\mathcal{S}(V_{k+1}))>\dim(\mathcal{S}(V_k))$ for each $0\le k\le K-1$.
4. $\mathcal{H}^l(\operatorname{sing}(\|V_k\|)\cap B_1)>0$ for each $0\le k\le K$.
5. $V_K=\boldsymbol{C}\times\mathbb{R}^m$ where $\boldsymbol{C}\backslash\{0\}$ is a smooth immersed cone for some $m\ge 0$.

In particular, the last two conditions imply $m=l$. Now, $\boldsymbol{C}$ becomes a stable cone with isolated singular point 0 of dimension $n-l$. By the classification of the stable cones, we know $n-l\ge 7$, and hence $l\le n-7$. This shows $\dim(\operatorname{sing}\|V\|\cap B_1)\le n-7$.

In the case $n=7$, we have

$$\mathcal{H}^\alpha(\operatorname{sing}\|V_K\|\cap B_1)=0$$

for any $\alpha>0$ by the above result.

If $\operatorname{sing}(\|V\|\cap B_1)$ is not discrete, then we can find $x_j\in\operatorname{sing}(\|V\|\cap B_1)$ such that $x_j\to x_0\in\operatorname{sing}(\|V\|\cap B_1)$. Now, up to a subsequence, we can assume $(\eta_{x_0,|x_j-x_0|})_\# V$ converges to $\boldsymbol{C}\in\operatorname{VarTan}(V,x_0)$ and we denote $\xi=\lim_{j\to\infty}\frac{x_j-x_0}{|x_j-x_0|}\neq 0$. So $\mathcal{S}(\boldsymbol{C})$ contains the line spanned by ξ. In particular, $\mathcal{H}^1(\operatorname{sing}(\|\boldsymbol{C}\|)\cap B_1)>0$ which is a contradiction. Hence, $\operatorname{sing}(\|V\|\cap B_1)$ is discrete. ◆

3.4 Proof of Bernstein Theorem

Now, we are ready to prove the Bernstein Theorem up to the dimension $n\le 7$.

Theorem 3.4.1

Suppose u is a solution of the minimal surface equation on $\mathbb{R}^n$ and $n\le 7$. Then, u is an affine function. ♡

More analysis on the stationary cones

Recall that the Jacobi operator is defined as

$$L(\varphi)=\Delta\varphi+|A|^2\varphi$$

for any smooth function φ on the regular part of M. If $M^n\subset\mathbb{R}^{n+1}$ is a stationary cone with isolated singular point 0, then we can rewrite the Jacobi operator as

$$L(\varphi)=\frac{\partial^2\varphi}{\partial r^2}+\frac{n-1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\left(\Delta^\Sigma\varphi+|A_\Sigma|^2\varphi\right)$$

where $\Sigma=M\cap\mathbb{S}^n$. This is called the link of the stationary cone, which is a minimal hypersurface in $\mathbb{S}^n$.

So it is important to study the operator $\Delta^\Sigma+|A_\Sigma|^2$ on the link Σ. Suppose Σ is not totally geodesic, we define $\lambda_1(\Sigma)$ to be the first eigenvalue of the operator $-\Delta^\Sigma-|A_\Sigma|^2$ by

$$\lambda_1(\Sigma)=\inf_{\varphi\in H^1(\Sigma),\varphi\neq 0}\frac{\int_\Sigma|\nabla\varphi|^2-|A_\Sigma|^2\varphi^2d\Sigma}{\int_\Sigma\varphi^2d\Sigma}$$

Theorem 3.4.2

We have $\lambda_1(\Sigma)\le-(n-1)$. ♡

Proof We use the following Simons inequality for the link Σ:

$$|A_\Sigma|\Delta^\Sigma|A_\Sigma| + |A_\Sigma|^4 \geq \frac{2}{n-1}|\nabla|A_\Sigma||^2 + (n-1)|A_\Sigma|^2.$$

Now, we directly integrate the inequality over Σ to get

$$\int_\Sigma |\nabla|A_\Sigma||^2 - |A_\Sigma|^4 \leq -(n-1)\int_\Sigma |A_\Sigma|^2.$$

Using $|A_\Sigma|$ as a test function in the Rayleigh quotient gives

$$\lambda_1(\Sigma) \leq \frac{\int_\Sigma |\nabla|A_\Sigma||^2 - |A_\Sigma|^4 \, d\Sigma}{\int_\Sigma |A_\Sigma|^2 \, d\Sigma} \leq -(n-1).$$

♦

Proposition 3.4.3

Suppose φ is a smooth function on Σ achieving the infimum in the definition of $\lambda_1(\Sigma)$. If u is a positive Jacobi field on M, then if we define

$$V(r) := \int_\Sigma \varphi(x)u(rx)d\Sigma(x)$$

Then

$$(V(r)r^{\kappa_n})' \leq 0$$

for $\kappa_n := \frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - (n-1)}$. ♠

Proof We compute

$$\begin{aligned} V''(r) &= \int_\Sigma \varphi(x)\frac{\partial^2}{\partial r^2}u(rx)\,d\Sigma(x) \\ &= -\frac{n-1}{r}V'(r) - \frac{1}{r^2}\int_\Sigma \varphi(x)\big(\Delta^\Sigma u(rx) + |A_\Sigma|^2 u(rx)\big)\,d\Sigma(x) \\ &= -\frac{n-1}{r}V'(r) - \frac{1}{r^2}\int_\Sigma u(rx)\big(\Delta^\Sigma \varphi(x) + |A_\Sigma|^2 \varphi(x)\big)\,d\Sigma(x) \\ &\leq -\frac{n-1}{r}V'(r) - \frac{n-1}{r^2}V(r) \end{aligned}$$

where in the last step we used that φ is an eigenfunction of the operator $-\Delta^\Sigma - |A_\Sigma|^2$ with eigenvalue $\lambda_1(\Sigma) \leq -(n-1)$, and that $u > 0$. The proof is finished by analyzing the resulting ODE inequality. ♦

We consider $W(t) = V(t^{-1/\beta})t^{\gamma/\beta}$.

Then, we can choose β and γ such that $W''(t) \leq 0$. To see this, we compute

$$W''(t) = \frac{1}{\beta^2}t^{\frac{\gamma}{\beta}-2}\left(V''(t^{-1/\beta})t^{-2/\beta} + V'(t^{-1/\beta})t^{-1/\beta}((1+\beta) - 2\gamma) + V(t^{-1/\beta})\gamma(\gamma-\beta)\right)$$

So we have

$$1 + \beta - 2\gamma = \gamma(\gamma - \beta) = n - 1$$

The choice of $\gamma = -\kappa_n$ and $\beta = 2\sqrt{\frac{(n-2)^2}{4} - (n-1)}$ satisfies the above equations, and hence $W''(t) \leq 0$. Since $t = r^{-\beta}$ and $W(t) = V(r)r^{\kappa_n}$, this concavity gives the desired monotonicity of $V(r)r^{\kappa_n}$ in the r variable.

We are ready to prove the Bernstein Theorem. We need to study the blow-down limit of the graph of u. This is due to De Giorgi [DG65].

Theorem 3.4.4

Let u solve the minimal surface equation on $\mathbb{R}^n$ for $n \leq 7$. Then the blow-down limit of the graph of u is a density-one hyperplane in $\mathbb{R}^{n+1}$. ♡

Proof Suppose M is the graph of u. Then it is a minimal hypersurface in $\mathbb{R}^{n+1}$, and it is minimizing area in $\mathbb{R}^{n+1}$.

Let P be the region $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y < u(x)\}$. By the previous result, we know P is perimeter minimizer in $\mathbb{R}^{n+1}$. We consider the blow-down limit of P defined as follows. Define $P_r := \frac{1}{r}P$. Up to a subsequence, P_r converges to a set P_∞ in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$. By compactness, P_∞ is a perimeter minimizer in $\mathbb{R}^{n+1}$. The boundary of P_r, denoted by M_r, converges to the boundary of P_∞ in the varifold sense; denote the limit by V_∞.

By the previous regularity result, we know $\operatorname{sing}\|V_\infty\|$ is empty or discrete. If it is empty, then V_∞ is a density-one hyperplane in $\mathbb{R}^{n+1}$. If it is discrete, then V_∞ can only have an isolated singular point at the vertex 0, since it is a stationary cone.

Now, we consider $P'_r := P_r + e_{n+1}$. This is again a perimeter minimizer in $\mathbb{R}^{n+1}$, and we have $P_r \subset P'_r$. Up to a subsequence, P'_r converges to a set P'_∞, and $P'_\infty = P_\infty + e_{n+1}$. In particular, $P_\infty \subset P'_\infty$. By the strong maximum principle, we have either $P_\infty = P'_\infty$, or their boundaries are disjoint. In the first case, we are done, since V_∞ is translation invariant along e_{n+1} direction, it can only be a density-one hyperplane in $\mathbb{R}^{n+1}$. In the second case, we know that $V_\infty + \lambda e_{n+1}$ is disjoint from V_∞ for any $\lambda > 0$. Hence, we can construct a positive Jacobi field u on M_∞ by $u := \langle \nu_{M_\infty}, e_{n+1}\rangle$ where $M_\infty := \operatorname{reg}\|V_\infty\|$. Now, for the function $V(r) := \int_\Sigma \varphi(x)u(rx)d\Sigma(x)$, we have

$$V(r) \leq \int_\Sigma \varphi(x) \leq C$$

where C is independent of r. On the other hand, we have $\kappa_7 = 2$, and hence

$$V(r) \geq \frac{V(1)}{r^2} \text{ for } 0 < r < 1,$$

which implies $V(r) \to +\infty$ as $r \to 0$. This contradiction implies Σ is totally geodesic, and hence M_∞ is a density-one hyperplane in $\mathbb{R}^{n+1}$. ♦

Proof of Bernstein Theorem

By Allard's regularity theorem, M_r converges smoothly to a minimal hypersurface M_∞ in $\mathbb{R}^{n+1}$.

In particular, if we denote A_r to be the second fundamental form of M_r, then for any fixed $x \in M = M_1$, we have

$$A_r(\frac{x}{r}) \to A_\infty(0) = 0$$

On the other hand, we have

$$A_r(\frac{x}{r}) = rA(x)$$

which implies $A(x) = 0$. Hence, M is flat, so u is an affine function. ♦

Chapter 4 Regularity and Compactness Theorems

Definition 4.0.1 (Regular and singular sets)

Given a smooth immersed hypersurface M in U (or just a subset of U), for any $x \in \bar{M} \cap U$, we say x is a ***regular point*** *of M if there exists a number $r > 0$ such that $B_r^{n+1}(x) \subset U$ and $\bar{M} \cap B_r^{n+1}(x)$ can be written as a union of finitely many smooth, compact, connected, embedded hypersurfaces Σ_i in $B_r^{n+1}(x)$ such that $\bar{\Sigma}_i \cap B_r^{n+1}(x) = \Sigma_i$. In general, we redefine M such that each point in M is a regular point and every regular point of M lies in M.*

The (interior) ***singular set*** *of M is defined by*

$$\mathrm{sing} M = (\bar{M} \backslash M) \cap U.$$

♣

4.1 Regularity and Compactness Results for Stable Minimal Hypersurfaces

The first result is the generalized Bernstein theorem from Schoen–Simon–Yau.

Theorem 4.1.1 (Schoen–Simon–Yau Bernstein Theorem, [SSY75])

Suppose $M \subset \mathbb{R}^{n+1}$ is a complete, stable, minimal hypersurface without boundary and with at most (intrinsic) Euclidean volume growth. Then, if $n \leq 5$, M must be an affine hyperplane.

♡

Remark 4.1.2. Note that the intrinsic Euclidean volume growth condition is weaker than the extrinsic Euclidean volume growth condition. This is also a key condition for the Stable Bernstein Theorem without area growth condition.

Theorem 4.1.3 (Schoen–Simon Regularity and Compactness, [SS81])

Let $\{M_k\}$ be a sequence of embedded, stable, orientable minimal hypersurfaces in $B_2^{n+1}(0)$ with the following properties:

1. $0 \in \bar{M}_k$ *for each* k.
2. $\mathcal{H}^{n-2}(\mathrm{sing} M_k) = 0$ *for each* k.
3. $\mathcal{H}^n(M_k \cap B_2^n(0)) \leq \Lambda$ *for some constant* $\Lambda > 0$ *independent of* k.

Then, up to a subsequence, M_k converges in the varifold sense to a stable minimal hypersurface M in $B_2^{n+1}(0)$, which is smooth except for a closed singular set of Hausdorff dimension at most $n-7$.

♡

In addition, for $n = 7$, the singular set is discrete.

Remark 4.1.4. M_k converges to a stable minimal hypersurface M in the varifold sense means that the varifold $|M_k|$ converges to the varifold $|M|$ in the sense of measures of weak limit.

Recall that a closed set S is of Hausdorff dimension at most k if for any $\varepsilon > 0$, we have $\mathcal{H}^{k+\varepsilon}(S) = 0$.

Corollary 4.1.5

Suppose $M \subset \mathbb{R}^{n+1}$ is a complete, stable, embedded minimal hypersurface without boundary and with at most extrinsic Euclidean volume growth. Then, if $n \leq 6$, M must be an affine hyperplane.

♠

Remark 4.1.6. The dimension $n \leq 6$ is sharp, as we already proved that Simons' cone is stable in $\mathbb{R}^8$.

Bellettini's work completes the corresponding area-growth statement in the borderline dimension $n = 6$ for stable immersions. To state the result in a form that is independent of whether one measures volume intrinsically or extrinsically, we also record the comparison theorem of Florit–Simon.

Theorem 4.1.7 (Florit–Simon intrinsic–extrinsic area equivalence, [FS26])

Let $\Sigma^d \hookrightarrow \mathbb{R}^N$ be a complete, connected, smooth minimal immersion, and let $p \in \Sigma$. Define the intrinsic and extrinsic area densities by

$$\mathbf{M}_R^{\mathrm{int}}(\Sigma, p) := \frac{|B_R^{\Sigma}(p)|}{R^d}, \qquad \mathbf{M}_R^{\mathrm{ext}}(\Sigma, p) := \frac{\mathrm{Area}(\Sigma \cap B_R^{\mathbb{R}^N}(p))}{R^d},$$

where $B_R^{\Sigma}(p) = \{x \in \Sigma : d_{\Sigma}(x, p) < R\}$, and the extrinsic area is counted with multiplicity. Then $\mathbf{M}_R^{\mathrm{int}}(\Sigma, p)$ and $\mathbf{M}_R^{\mathrm{ext}}(\Sigma, p)$ are monotone nondecreasing in R, and

$$\lim_{R\to\infty} \mathbf{M}_R^{\mathrm{int}}(\Sigma, p) = \lim_{R\to\infty} \mathbf{M}_R^{\mathrm{ext}}(\Sigma, p) \in [\omega_d, \infty],$$

where $\omega_d = |B_1^{\mathbb{R}^d}|$. In particular, Σ has bounded intrinsic area density if and only if it has bounded extrinsic area density; in either case, the immersion is proper. ♡

Corollary 4.1.8 (Area-growth stable Bernstein for immersions, [SSY75, Bel25, FS26])

Let $2 \le n \le 6$, and let $\Sigma^n \hookrightarrow \mathbb{R}^{n+1}$ be a complete, connected, two-sided, stable minimal immersion. If Σ has Euclidean area growth, equivalently

$$|B_R^{\Sigma}(p)| \le \Lambda R^n \quad \textit{for some } p \in \Sigma,\ \Lambda < \infty,\ \textit{and all } R > 0,$$

or

$$\mathrm{Area}(\Sigma \cap B_R^{\mathbb{R}^{n+1}}(p)) \le \Lambda R^n \quad \textit{for some } p \in \mathbb{R}^{n+1},\ \Lambda < \infty,\ \textit{and all } R > 0,$$

then Σ is an affine hyperplane. ♠

Remark 4.1.9. For $2 \le n \le 5$, this is the Schoen–Simon–Yau area-growth stable Bernstein theorem. The new borderline case is $n = 6$: Bellettini proves the classification under extrinsic Euclidean area growth, and Theorem 4.1.7 converts intrinsic area growth into the same extrinsic hypothesis. Thus the area-growth version of the immersed stable Bernstein theorem is settled in the full sharp range $2 \le n \le 6$.

A natural question is whether the compactness conclusion in Theorem 4.1.3 remains true if we assume $\mathcal{H}^{n-1}(\mathrm{sing} M_k) = 0$ instead of $\mathcal{H}^{n-2}(\mathrm{sing} M_k) = 0$. The answer is yes, by the following deep result of Wickramasekera.

Theorem 4.1.10 (Wickramasekera's Regularity Theorem, [Wic14])

Suppose V_i is a sequence of stationary integral n-varifolds in $B_2^{n+1}(0)$ and V_i also satisfies the following conditions:

1. $0 \in \mathrm{spt}\|V_i\|$.
2. $\|V_i\|(B_2^{n+1}(0)) \le \Lambda$ *for some constant $\Lambda > 0$ independent of i.*
3. *(Stability) Each V_i is stable in $B_2^{n+1}(0)$ on its regular set, i.e., for any $\phi \in C_c^1(\mathrm{reg} V_i)$,*
$$\int_{\mathrm{reg} V_i} |A_i|^2 \phi^2 d\|V_i\| \le \int_{\mathrm{reg} V_i} |\nabla \phi|^2 d\|V_i\|.$$
4. *(Alpha-Structural Hypothesis) There exists $\alpha \in (0, 1)$ such that for each i, no point of $\mathrm{spt}\|V_i\| \cap B_1^{n+1}(0)$ has a neighborhood in which $\mathrm{spt}\|V_i\|$ is the union of three or more embedded $C^{1,\alpha}$ hypersurfaces-with-boundary meeting only along their common boundary.*

Then, up to a subsequence, V_i converges in the varifold sense to a stationary integral n-varifold V_∞ in $B_2^{n+1}(0)$, which is stable and whose singular set in $B_2^{n+1}(0)$ has Hausdorff dimension at most $n-7$.

Remark 4.1.11. The theorem is formulated for stationary integral varifolds, so no orientability assumption is part of the statement. In applications to stable immersions, two-sidedness is imposed separately when one writes the stability inequality on the regular set.

The next conjecture concerns compactness for stable minimal immersions with singular sets.

Conjecture 4.1.12

The class of branched two-sided stable minimal n-dimensional immersions with the singular set of locally finite $(n-2)$-measure is compact under varifold convergence.

The first result in this direction is the following density-2 regularity theorem.

Theorem 4.1.13 (Density-2 regularity)

Let $\delta \in (0,1)$. Suppose M_k is a sequence of orientable stable minimal hypersurfaces immersed in $B_2^{n+1}(0)$ such that:

1. $0 \in \bar{M}_k$.
2. $\|M_k\|(B_2^{n+1}(0)) \leq (3-\delta)\omega_n 2^n$.
3. $\mathcal{H}^{n-2}(\mathrm{sing} M_k) = 0$.

Then, up to a subsequence, M_k converges in the varifold sense to a stable minimal hypersurface M in $B_2^{n+1}(0)$, which is smooth except for a closed singular set of Hausdorff dimension at most $n-7$.

The general case, with non-optimal singular set dimension, is proved by Hong-Li-Wang [HLW24].

Theorem 4.1.14 ($\dim_{\mathcal{H}}(\mathrm{sing} M) < n-4+\frac{4}{n}$ regularity, [HLW24])

Suppose M_k is a sequence of orientable stable minimal hypersurfaces immersed in $B_2^{n+1}(0)$ such that:

1. $0 \in \bar{M}_k$.
2. $\|M_k\|(B_2^{n+1}(0)) \leq \Lambda$.
3. $\sup_k \dim_{\mathcal{H}}(\mathrm{sing} M_k) < n-4+\frac{4}{n}$.

Then, up to a subsequence, M_k converges in the varifold sense to a stable minimal hypersurface M in $B_2^{n+1}(0)$, which is smooth except for a closed singular set of Hausdorff dimension at most $n-7$.

Minter–Xiao subsequently proved the optimal non-branched version of this regularity and compactness theorem.

Theorem 4.1.15 (Minter–Xiao optimal immersed regularity, [MX26])

Let $n \geq 2$. Suppose M_k is a sequence of two-sided stable minimal hypersurfaces smoothly and properly immersed in $B_1^{n+1}(0)$ such that:

1. $0 \in \bar{M}_k$ *for each* k;
2. $\sup_k \mathcal{H}^n(M_k \cap B_1^{n+1}(0)) < \infty$;
3. $\mathcal{H}^{n-2}(\mathrm{sing} M_k) = 0$ *for each* k*, where*

$$\mathrm{sing} M_k := B_1^{n+1}(0) \cap (\bar{M}_k \setminus M_k).$$

Then, after passing to a subsequence, M_k converges as varifolds to a stationary integral varifold V in $B_1^{n+1}(0)$. Moreover, there is a relatively closed set

$$S \subset \operatorname{spt}\|V\| \cap B_1^{n+1}(0), \qquad \dim_{\mathcal{H}} S \le n-7,$$

such that V is represented on $B_1^{n+1}(0) \setminus S$ by a proper, two-sided, stable minimal immersion, and M_k converges locally smoothly to this immersion away from S. In particular, $S = \emptyset$ for $2 \le n \le 6$, while S is discrete for $n = 7$. ♡

Remark 4.1.16. The hypothesis $\mathcal{H}^{n-2}(\operatorname{sing} M_k) = 0$ is the optimal non-branched assumption. It improves Theorem 4.1.14, where the non-immersed singular set is required to have Hausdorff dimension strictly smaller than $n - 4 + \frac{4}{n}$. The branch point case is not included here: when the non-immersed singular set has positive $\mathcal{H}^{n-2}$-measure, branch points may occur and the corresponding compactness theory remains a separate problem.

4.2 Key Steps in the Regularity Proofs

Up to a subsequence, we can assume $|M_k|$ or V_k converges to V in the varifold sense. Pick any point $x \in \operatorname{spt}\|V\|$. We need to analyze the tangent cone of V at x. There are several cases:

- Hyperplanes. We need to develop a sheeting theorem to show the regularity and smooth convergence.
- Classical Cones. We need a minimum distance theorem (embedded case) or a decomposition theorem (immersed case) to show the regularity and smooth convergence.
- $\boldsymbol{C} \times \mathbb{R}^m$ where $\boldsymbol{C}$ is a stable cone with isolated singularity. Classification of stable cones.
- other cones. We need dimension reduction argument to reduce to the previous case.

4.3 Regularity in the Immersed Setting

One of the key ingredients in the proof of Theorem 4.1.14 is the following ε-regularity theorem.

Theorem 4.3.1 (ε-regularity for $|A|$)

Let $n \ge 3$. Suppose M^n is a two-sided stable minimal hypersurface immersed in $B_4^{n+1}(0)$ and the singular set of M satisfies $\bar{n} := \dim(\operatorname{sing} M) < n - 2 - \frac{2(n-2)}{n}$. Additionally, assume $\mathcal{H}^n(M \cap B_4^{n+1}(0)) \le \Lambda$ for some $\Lambda \in (0, +\infty)$. Then, for any $\alpha \in \left(\frac{n-2}{n}, \min\left\{\frac{n-\bar{n}-2}{2}, 1\right\}\right)$, there exists $\varepsilon = \varepsilon(n, \bar{n}, \alpha, \Lambda) \in (0,1)$ such that if

$$\int_{B_2^{n+1}(0) \cap M} |A|^{2\alpha} \le \varepsilon,$$

then

$$\sup_{B_{\frac{1}{2}}^{n+1}(0) \cap M} |A|^{2\alpha} \le C \int_{B_2^{n+1}(0) \cap M} |A|^{2\alpha}$$

for some constant $C = C(n, \bar{n}, \Lambda, \alpha)$. ♡

The above result relies on the following weak (intrinsic) Caccioppoli inequality.

Lemma 4.3.2

For any $\alpha \in (\frac{n-2}{n}, \min\left\{\frac{n-\bar{n}-2}{2}, 1\right\})$*, and any locally Lipschitz function* ϕ *supported in* $B_3^{n+1}(0)$*, we have*

$$\int_{M\cap\{u>k\}} \left(1-\frac{k}{u}\right)|\nabla u|^2\phi^2 \le C\int_{M\cap\{u>k\}}(u-k)^2|\nabla\phi|^2 + Ck^2\int_{M\cap\{u>k\}}\left((u-k)^{\frac{2}{\alpha}}+k^{\frac{2}{\alpha}}\right)\phi^2, \tag{4.3.1}$$

where the constant $C = C(n,\bar{n},\Lambda,\alpha)$*. Here* $u = |A|^\alpha$*.* ♠

Proof We first show that (4.3.1) holds for bounded locally Lipschitz ϕ with compact support in $B_4^{n+1}(0)$, vanishing in a neighborhood of $\operatorname{sing}M \cap B_4^{n+1}(0)$, and for any $\alpha \in (\frac{n-2}{n}, 1)$.

For such ϕ and α, choose $\varphi = (|A|^\alpha - k)^+\phi$ for $k \ge 0$. One checks that $((|A|^\alpha - k)^+)^2 \in C^1(M) \cap W^{2,\infty}_{\mathrm{loc}}(M)$. Hence we can insert φ into the stability inequality.

We observe that

$$\begin{aligned}\frac{1}{2}\Delta(|A|^\alpha-k)^2 &= \alpha\left(1-\frac{k}{|A|^\alpha}\right)|A|^{2\alpha-2}|A|\Delta|A| \\ &\quad+\alpha\left(\left(1-\frac{k}{|A|^\alpha}\right)(\alpha-1)+\alpha\right)|A|^{2\alpha-2}|\nabla|A||^2.\end{aligned}$$

Using Simons' inequality

$$|A|\Delta|A| \ge \frac{2}{n}|\nabla|A||^2 - |A|^4,$$

we obtain

$$\begin{aligned}\int_M|\nabla\varphi|^2 &= \int_{M_{>k}}\alpha^2|A|^{2\alpha-2}|\nabla|A||^2\phi^2+(|A|^\alpha-k)^2|\nabla\phi|^2+\frac{1}{2}\left\langle\nabla(|A|^\alpha-k)^2,\nabla\phi^2\right\rangle \\ &= \int_{M_{>k}}\alpha^2|A|^{2\alpha-2}|\nabla|A||^2\phi^2+(|A|^\alpha-k)^2|\nabla\phi|^2-\frac{1}{2}\phi^2\Delta(|A|^\alpha-k)^2 \\ &= \int_{M_{>k}}(|A|^\alpha-k)^2|\nabla\phi|^2-\int_{M_{>k}}\alpha\left(1-\frac{k}{|A|^\alpha}\right)|A|^{2\alpha-2}|A|\Delta|A|\phi^2 \\ &\quad+\alpha(1-\alpha)\left(1-\frac{k}{|A|^\alpha}\right)|A|^{2\alpha-2}|\nabla|A||^2\phi^2 \\ &\le \int_{M_{>k}}((|A|^\alpha-k)^2)|\nabla\phi|^2 \\ &\quad-\int_{M_{>k}}\alpha\left(\frac{2}{n}+\alpha-1\right)\left(1-\frac{k}{|A|^\alpha}\right)|A|^{2\alpha-2}|\nabla|A||^2\phi^2+\int_{M_{>k}}\alpha|A|^{2\alpha+2}\phi^2\end{aligned}$$

where $M_{>k}$ denotes $M\cap\{|A|^\alpha>k\}$. On the other hand, by stability, we have

$$\int_M|\nabla\varphi|^2 \ge \int_M|A|^2\varphi^2 = \int_{M_{>k}}|A|^2(|A|^\alpha-k)^2\phi^2.$$

Now, we write $\delta := \alpha - \frac{n-2}{n} > 0$. Then the stability inequality gives

$$\begin{aligned}&\delta\int_{M_{>k}}\alpha\left(1-\frac{k}{|A|^\alpha}\right)|A|^{2\alpha-2}|\nabla|A||^2\phi^2 \\ &\le \int_{M_{>k}}((|A|^\alpha-k)^2)|\nabla\phi|^2+\int_{M_{>k}}\alpha|A|^{2\alpha}|A|^2\phi^2 \\ &\quad-\int_{M_{>k}}|A|^2(|A|^\alpha-k)^2\phi^2.\end{aligned}$$

Now, let $u = |A|^\alpha$. Then,

$$\frac{\delta}{\alpha}\int_{M_{>k}}\left(1-\frac{k}{u}\right)|\nabla u|^2\phi^2 \le \int_{M_{>k}}(u-k)^2|\nabla\phi|^2 + \int_{M_{>k}} u^{\frac{2}{\alpha}}\left(\alpha u^2-(u-k)^2\right)\phi^2. \tag{4.3.2}$$

Now we estimate $u^{\frac{2}{\alpha}}\left(\alpha u^2-(u-k)^2\right)$.

$$\alpha u^2-(u-k)^2 = -(1-\alpha)(u-k)^2+2\alpha k(u-k)+\alpha k^2 \le \frac{\alpha^2k^2}{1-\alpha}+\alpha k^2 = \frac{\alpha}{1-\alpha}k^2.$$

where we used Young's inequality. By the trivial inequality $(x+y)^a \le 2^a(x^a+y^a)$ for $x,y\ge 0$, $a>0$, we have

$$u^{\frac{2}{\alpha}}\left(\alpha u^2-(u-k)^2\right) \le 2^{\frac{2}{\alpha}}\frac{\alpha}{1-\alpha}\left((u-k)^{\frac{2}{\alpha}}+k^{\frac{2}{\alpha}}\right)k^2.$$

Substituting this into (4.3.2), we obtain (4.3.1).

To complete the proof, we show by approximation that (4.3.1) holds for any bounded locally Lipschitz ϕ supported in $B_3(0)$, assuming $\alpha \in \left(\frac{n-2}{n}, \min\left\{\frac{n-\bar{n}-2}{2}, 1\right\}\right)$. Note that ϕ may be non-zero on the singular set of M.

We first derive a preliminary estimate on $|A|$.

Lemma 4.3.3

If $\mathcal{H}^{n-2}(\operatorname{sing}(M)\cap B_4^{n+1}(0)) = 0$, then we have $|A| \in L^2(B_{\frac{7}{2}}^{n+1}(0)\cap M)$ and

$$\int_{M\cap B_\rho^{n+1}(x)}|A|^2 \le C\rho^{n-2},$$

for any $x \in B_{\frac{7}{2}}^{n+1}(0)$ and $\rho \in (0,\frac{1}{4})$, where $C = C(\Lambda)$.

Proof For each $\varepsilon > 0$, we choose balls $\left\{B_{r_i}^{n+1}(x_i)\right\}_{i=1}^N$ such that $\operatorname{sing}(M)\cap B_4^{n+1}(0) \subset \bigcup_{i=1}^N B_{r_i}^{n+1}(x_i)$ and $\sum_{i=1}^N r_i^{n-2} \le \varepsilon$. We choose ζ_i to be a non-negative C^1 function such that ζ_i is supported outside of $B_{r_i}^{n+1}(x_i)$, $\zeta_i = 1$ outside of $B_{2r_i}^{n+1}(x_i)$, and $|\nabla\zeta_i| \le \frac{2}{r_i}$. Then, we define $\zeta_\varepsilon = \min_{1\le i\le N}\zeta_i$. We insert $\zeta_\varepsilon\phi$ into the stability inequality where ϕ is a non-negative locally Lipschitz function with compact support in $B_4^{n+1}(0)$. Then,

$$\int_{M\cap B_4^{n+1}(0)}|A|^2\phi^2\zeta_\varepsilon^2 \le 2\int_{M\cap B_4^{n+1}(0)}|\nabla\phi|^2\zeta_\varepsilon^2 + 2\int_{M\cap B_4^{n+1}(0)}|\nabla\zeta_\varepsilon|^2\phi^2$$

by Cauchy-Schwarz inequality. Note that

$$\begin{aligned}\int_{M\cap B_4^{n+1}(0)}|\nabla\zeta_\varepsilon|^2\phi^2 &\le \|\phi\|_{L^\infty(B_4^{n+1}(0))}^2\sum_{i=1}^N\int_{M\cap B_{2r_i}^{n+1}(x_i)}|\nabla\zeta_i|^2\\ &\le C\|\phi\|_{L^\infty(B_4^{n+1}(0))}^2\sum_{i=1}^N r_i^{n-2} \le C\|\phi\|_{L^\infty(B_4^{n+1}(0))}^2\varepsilon\end{aligned}$$

which converges to 0 as $\varepsilon \to 0^+$. Then, we have

$$\int_M|A|^2\phi^2 \le 2\int_M|\nabla\phi|^2.$$

In particular, it implies $|A| \in L^2(B_{\frac{7}{2}}^{n+1}(0))$ if we choose $\phi \equiv 1$ on $B_{\frac{7}{2}}^{n+1}(0)$. Now, we choose ϕ supported on $B_{2\rho}^{n+1}(x)$, and equal to 1 on $B_\rho^{n+1}(x)$, with $|\nabla\phi| \le \frac{2}{\rho}$. Together with the monotonicity formula, we have

$$\int_{M\cap B_\rho^{n+1}(x)}|A|^2 \le 2\int_{M\cap B_{2\rho}^{n+1}(x)\setminus B_\rho^{n+1}(x)}\frac{1}{\rho^2} \le C\rho^{n-2},$$

for some $C = C(\Lambda)$.

The remaining part is similar to the proof of the previous lemma. Based on the assumption of $\bar{n}$ and α, we know $\mathcal{H}^{n-2-2\alpha}(\text{sing}M) = 0$. Therefore, for any $\varepsilon > 0$, there exist $B^{n+1}_{r_1}(x_1), B^{n+1}_{r_2}(x_2), \cdots, B^{n+1}_{r_N}(x_N)$ with $x_i \in B^{n+1}_{\frac{7}{2}}(0)$ and $0 < r_i < \frac{1}{4}$ for each $1 \le i \le N$, such that

$$\text{sing}M \cap B^{n+1}_3(0) \subset \bigcup_{i=1}^N B^{n+1}_{r_i}(x_i), \quad \text{and} \quad \sum_{i=1}^N r_i^{n-2-2\alpha} \le \varepsilon. \tag{4.3.3}$$

We choose ζ_i and ζ_ε as in the proof of Lemma 4.3.3. For any locally Lipschitz ϕ with compact support in $B^{n+1}_3(0)$, $\zeta_\varepsilon\phi$ vanishes near $\text{sing}M$, allowing us to use (4.3.1) with $\zeta_\varepsilon\phi$ in place of ϕ. Thus, we have

$$\begin{aligned}&\int_{M_{>k}} \left(1 - \frac{k}{u}\right)|\nabla u|^2\zeta_\varepsilon^2\phi^2 \\ &\le C\int_{M_{>k}} (u-k)^2\zeta_\varepsilon^2|\nabla\phi|^2 + Ck^2\int_{M_{>k}} \left((u-k)^{\frac{2}{\alpha}} + k^{\frac{2}{\alpha}}\right)\zeta_\varepsilon^2\phi^2 + C\int_{M_{>k}} (u-k)^2|\nabla\zeta_\varepsilon|^2\phi^2,\end{aligned} \tag{4.3.4}$$

by the Cauchy-Schwarz inequality.

For the first two terms on the right-hand side of (4.3.4), since $|A|^{2\alpha}$ and $|A|^2$ are integrable in $B^{n+1}_3(0)\cap M$ by Lemma 4.3.3, we can let $\varepsilon \to 0^+$, leading to

$$C\int_{M_{>k}} (u-k)^2|\nabla\phi|^2 + Ck^2\int_{M_{>k}} \left((u-k)^{\frac{2}{\alpha}} + k^{\frac{2}{\alpha}}\right)\phi^2.$$

Then, we need to show

$$\lim_{\varepsilon\to 0^+}\int_{M_{>k}} (u-k)^2|\nabla\zeta_\varepsilon|^2\phi^2 = 0.$$

Applying Lemma 4.3.3, (4.3.3), and Hölder inequality, we obtain

$$\begin{aligned}&\int_{M_{>k}} (u-k)^2|\nabla\zeta_\varepsilon|^2\phi^2 \\ &\le \sum_{i=1}^N \|\phi\|^2_{L^\infty(B^{n+1}_3(0))}\int_{M\cap B^{n+1}_{r_i}(x_i)} |A|^{2\alpha}|\nabla\zeta_i|^2 \\ &\le \|\phi\|^2_{L^\infty(B^{n+1}_3(0))}\sum_{i=1}^N\left(\int_{M\cap B^{n+1}_{r_i}(x_i)}|A|^2\right)^\alpha\left(\int_{M\cap B^{n+1}_{r_i}(x_i)}|\nabla\zeta_i|^{\frac{2}{1-\alpha}}\right)^{1-\alpha} \\ &\le C\|\phi\|^2_{L^\infty(B^{n+1}_3(0))}\sum_{i=1}^N r_i^{\alpha(n-2)}r_i^{(n-\frac{2}{1-\alpha})(1-\alpha)} = C\|\phi\|^2_{L^\infty(B^{n+1}_3(0))}\sum_{i=1}^N r_i^{n-2-2\alpha} \\ &\le C\|\phi\|^2_{L^\infty(B^{n+1}_3(0))}\varepsilon.\end{aligned}$$

Hence, letting $\varepsilon \to 0^+$, we conclude that (4.3.1) holds for any bounded locally Lipschitz ϕ supported in $B^{n+1}_3(0)$. ◆

Remark 4.3.4. The preceding proof also shows that $|\nabla u|^2$ is integrable in $B^{n+1}_3(0)\cap M$ and hence $u \in W^{1,2}(B^{n+1}_3(0)\cap M)$.

We now prove Theorem 4.3.1.

Proof of Theorem 4.3.1 Consider

$$k_l = d\left(1 - \frac{1}{2^{l-1}}\right), \quad \text{and} \quad R_l = \frac{1}{2} + \frac{1}{2^l},$$

for $0 < d \le 1$. k_l increases to d and R_l decreases to $1/2$ as $l \to \infty$. For simplicity, we write $\Omega_l = M \cap \{u > k_l\} \cap B^{n+1}_{R_l}(0)$.

Applying the previous lemma and noting that

$$1 - \frac{k_l}{u} \ge 1 - \frac{k_l}{k_{l+1}} \ge \frac{1}{2^l},$$

for any $u > k_{l+1}$, we have

$$\frac{1}{2^l}\int_{M_{>k_{l+1}}}|\nabla u|^2\phi^2 \le C\Bigg[\int_{M_{>k_l}}(u-k_l)^2|\nabla\phi|^2 \\ +d^2\int_{M_{>k_l}}(u-k_l)^{\frac{2}{\alpha}}\phi^2 + d^{2+\frac{2}{\alpha}}\int_{M_{>k_l}}\phi^2\Bigg].$$

Using

$$|\nabla((u-k_{l+1})\phi)|^2 \le 2|\nabla u|^2\phi^2 + 2(u-k_{l+1})^2|\nabla\phi|^2,$$

we obtain

$$\int_{M_{>k_{l+1}}}|\nabla((u-k_{l+1})\phi)|^2 \\ \le 2^lC\left[\int_{M_{>k_l}}(u-k_l)^2|\nabla\phi|^2 + d^2\int_{M_{>k_l}}(u-k_l)^{\frac{2}{\alpha}}\phi^2 + d^{2+\frac{2}{\alpha}}\int_{M_{>k_l}}\phi^2\right].$$

Now choose ϕ supported in $B^{n+1}_{R_l}(0)$, with $\phi = 1$ on $B^{n+1}_{R_{l+1}}(0)$, $|\nabla\phi| \le 2^{l+2}$, and $0 \le \phi \le 1$. Together with the Michael–Simon inequality [MS73]

$$\left(\int_M|\varphi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \le C\int_M|\nabla\varphi|^2,$$

for a constant C only depending on n. Then, we have

$$\left(\int_{\Omega_{l+1}}(u-k_{l+1})^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ \le C^l\left[\int_{\Omega_l}(u-k_l)^2 + d^2\int_{\Omega_l}(u-k_l)^{\frac{2}{\alpha}} + d^{2+\frac{2}{\alpha}}\mathcal{L}^n(\Omega_l)\right]. \tag{4.3.5}$$

Using the fact that when $u \ge k_l$, we know $u - k_{l-1} \ge \frac{d}{2^{l-1}}$. Hence, for any $0 \le \beta \le \frac{2n}{n-2}$,

$$\int_{\Omega_l}(u-k_l)^\beta \le \int_{\Omega_l}(u-k_l)^\beta\left(\frac{2^{l-1}}{d}\right)^{\frac{2n}{n-2}-\beta}(u-k_{l-1})^{\frac{2n}{n-2}-\beta} \\ \le \frac{C^l}{d^{\frac{2n}{n-2}-\beta}}\int_{\Omega_{l-1}}(u-k_{l-1})^{\frac{2n}{n-2}},$$

where constant $C = C(n)$. Note that since $\frac{2}{\alpha} < \frac{2n}{n-2}$, we can use the above inequality with $\beta = 0, 2$, and $\frac{2}{\alpha}$ in (4.3.5) to obtain

$$S_{l+1}^{\frac{n-2}{n}} \le C^l\left(\frac{1}{d^{\frac{2n}{n-2}-2}} + \frac{1}{d^{\frac{2n}{n-2}-\frac{2}{\alpha}-2}} + \frac{1}{d^{\frac{2n}{n-2}-2-\frac{2}{\alpha}}}\right)S_{l-1}, \tag{4.3.6}$$

where

$$S_l := \int_{\Omega_l}(u-k_l)^{\frac{2n}{n-2}}.$$

Using $d \le 1$, (4.3.6) implies

$$\frac{S_{l+1}}{d^{\frac{2n}{n-2}}} \le C^l\left(\frac{S_{l-1}}{d^{\frac{2n}{n-2}}}\right)^{\frac{n}{n-2}}, \tag{4.3.7}$$

for some $C = C(n, \bar{n}, \Lambda, \alpha)$. Iterating (4.3.7), we obtain

$$\frac{S_{2l+1}}{d^{\frac{2n}{n-2}}} \le C^{2+\frac{4(n-2)}{n}+\cdots+2l(\frac{n-2}{n})^{l-1}}\left(C^2\frac{S_1}{d^{\frac{2n}{n-2}}}\right)^{(\frac{n}{n-2})^l} \le C^{\frac{n^2}{2}}\left(C^2\frac{S_1}{d^{\frac{2n}{n-2}}}\right)^{(\frac{n}{n-2})^l}.$$

Hence, if we require

$$S_1 \le (\varepsilon' d)^{\frac{2n}{n-2}},$$

for some positive ε' only depending on n, $\bar{n}$, δ, Λ, and α, then we have $\lim_{l\to\infty} S_{2l+1} = 0$. This implies

$$|A|^\alpha \le d \text{ on } B^{n+1}_{\frac{1}{2}}(0).$$

Finally, we need to ensure $S_1 \le (\varepsilon' d)^{\frac{2n}{n-2}}$.

Using Lemma 4.3.2 with $k = 0$ and a suitable test function, we obtain

$$\int_{M\cap B^{n+1}_{\frac{3}{2}}(0)} |\nabla u|^2 \le C \int_{M\cap B^{n+1}_2(0)} u^2$$

for some $C = C(n, \bar{n}, \Lambda, \alpha)$. Thus, by Michael–Simon's inequality, we have

$$S_1^{\frac{n-2}{n}} \le C \int_{M\cap B^{n+1}_1(0)} |\nabla(u\varphi)|^2 \le C \int_{M\cap B^{n+1}_2(0)} u^2 = C \int_{M\cap B^{n+1}_2(0)} |A|^{2\alpha}. \tag{4.3.8}$$

for φ supported on $B^{n+1}_{\frac{3}{2}}(0)$, equal to 1 on $B^{n+1}_1(0)$, and $|\nabla\varphi| \le 4$, where $C = C(n, \bar{n}, \Lambda, \alpha)$. Now choose $\varepsilon = \frac{(\varepsilon')^2}{C}$, where C is the constant in (4.3.8), and set $d = \sqrt{\frac{1}{\varepsilon}\int_{M\cap B^{n+1}_2(0)} |A|^{2\alpha}} \in (0, 1]$ by assumption. Consequently, $S_1^{\frac{n-2}{n}} \le (\varepsilon' d)^2$ holds by (4.3.8). For such a choice of d, we know $\lim_{l\to\infty} S_l = 0$, which implies

$$\sup_{B^{n+1}_{\frac{1}{2}}(0)\cap M} |A|^{2\alpha} \le d^2 = C \int_{M\cap B^{n+1}_2(0)\cap M} |A|^{2\alpha}$$

for some $C = C(n, \bar{n}, \Lambda, \alpha)$. ♦

With this ε-regularity theorem, we can prove the following results.

Proposition 4.3.5

Let $n \ge 3$, and $\bar{n} < n - 4 + \frac{4}{n}$. Suppose M_j is a sequence of immersed, two-sided, stable minimal hypersurfaces in $B^{n+1}_4(0)$ with $\dim(\operatorname{sing}(M_j)\cap B^{n+1}_4(0)) \le \bar{n}$, and that M_j converges (as varifolds) to $q|P\cap B^{n+1}_4(0)|$ as $j\to\infty$, where P is a hyperplane and q is a positive integer. Then,

$$\lim_{j\to\infty} \sup_{B^{n+1}_{\frac{1}{2}}(0)\cap M_j} |A_{M_j}| = 0.$$

Moreover, $\operatorname{sing} M_j \cap B^{n+1}_{\frac{1}{4}}(0) = \emptyset$ and $M_j \cap B^{n+1}_{\frac{1}{4}}(0)$ has exactly q connected components for j large enough, and each component of $M_j\cap B^{n+1}_{\frac{1}{4}}(0)$ converges smoothly to P in $B^{n+1}_{\frac{1}{4}}(0)$ as smooth immersions. ♠

Proof We suppose $P = \{x_{n+1} = 0\}$.

We claim that

$$\int_{M_j\cap B_2(0)} |A_j|^{2\alpha} \to 0.$$

We also use the following theorem due to Schoen–Simon [SS81].

Theorem 4.3.6 (Schoen–Simon)

We have the following inequality

$$\int |A|^2\varphi^2 \le C \int |\nabla\varphi|^2(1 - (\nu\cdot e_{n+1})^2)$$

♡

By the monotonicity formula, we have

$$\lim_{j\to\infty} \sup_{B^{n+1}_3(0)\cap M_j} |x_{n+1}| = 0.$$

Otherwise, we can find a sequence $p_j \in M_j$ such that $|p_{j,n+1}| \ge \delta > 0$ for some $0 < \delta < 1$. By the monotonicity formula, we have

$$\mathcal{H}^n(M_j \cap B^{n+1}_{\frac{\delta}{2}}(p_j)) \geq C\delta^n.$$

Then, we have

$$q|P \cap B^{n+1}_4(0)|(B_\delta(p_0)) \geq \limsup_{j\to\infty} V_j(B_{\frac{\delta}{2}}(p_j)) \geq C\delta^n,$$

where $|p_{0,n+1}| \geq \delta$, a contradiction.

Now, we choose $\varphi^2 x_{n+1}e_{n+1}$ as a test vector field in the first variation formula where φ is a smooth function supported in $B^{n+1}_3(0)$. Then, we have

$$\int \varphi^2|e^\top_{n+1}|^2 = -2\int x_{n+1}\varphi\nabla\varphi \cdot e^\top_{n+1} \leq \frac{1}{2}\int \varphi^2|e^\top_{n+1}|^2 + 2\int |\nabla\varphi|^2|x_{n+1}|^2.$$

So we have

$$\int_{M_j} \varphi^2(1-(\nu\cdot e_{n+1})^2) \leq 4\int_{M_j} |\nabla\varphi|^2|x_{n+1}|^2,$$

which goes to zero as $j\to\infty$. Hence

$$\int_{M_j\cap B_2(0)} |A_j|^2 \to 0.$$

Now, we use Hölder's inequality to get

$$\int_{M_j\cap B_2(0)} |A_j|^{2\alpha} \leq \left(\int_{M_j\cap B_2(0)} |A_j|^2\right)^\alpha (\mathcal{H}^n(M_j\cap B_2(0)))^{1-\alpha} \to 0.$$

Now we apply the ε-regularity theorem (Theorem 4.3.1) to get

$$\lim_{j\to\infty} \sup_{B^{n+1}_{\frac{1}{2}}(0)\cap M_j} |A_{M_j}| = 0. \tag{4.3.9}$$

Next, let us denote $S_j = P(\mathrm{sing}(M_j))$ where P is the projection to $\{x_{n+1}=0\}$. Then the projection P gives a covering map from $M_j\cap(B^n_{\frac{1}{4}}(0)\backslash S_j)\times\mathbb{R}$ to $(B^n_{\frac{1}{4}}(0)\backslash S_j)\times\{0\}$, and the covering degree is q by (4.3.9) for j large enough. Since $B^n_{\frac{1}{4}}(0)\backslash S_j$ is simply connected (because $\dim(S_j)\leq\bar{n}$), we know that $M_j\cap(B^n_{\frac{1}{4}}(0)\setminus S_j)\times\mathbb{R}$ has exactly q connected components, and each component can be written as a graph of a smooth function over $B^n_{\frac{1}{4}}(0)\backslash S_j$. By the removable singularity theorem (cf. [DGS65, Sim77]), we know that such a function can be extended to a smooth function on $B^n_{\frac{1}{4}}$ which solves the minimal surface equation. Hence, $\mathrm{sing} M_j\cap B^n_{\frac{1}{4}}(0)\times\mathbb{R}=\emptyset$ and it can be decomposed into q connected components, each of which converges smoothly to $B^n_{\frac{1}{4}}(0)\times\{0\}$ as smooth immersions by standard PDE theory. ◆

We consider a flat cone $\boldsymbol{C}$ in $\mathbb{R}^{n+1}$ defined as a union of hyperplanes and half-hyperplanes. Explicitly, we write

$$\boldsymbol{C} := \sum_{i=1}^{N_1} p_i|P_i| + \sum_{i=1}^{N_2} q_i|H_i|,$$

for $\{p_i\}_{i=1}^{N_1}$, $\{q_i\}_{i=1}^{N_2} \subset \mathbb{N}$. Here $\{P_i\}_{i=1}^{N_1}$ are distinct hyperplanes and $\{H_i\}_{i=1}^{N_2}$ are distinct half-hyperplanes such that $0\in P_i$ for each $1\leq i\leq N_1$, $0\in\bar{H}_i$ for each $1\leq i\leq N_2$, and $H_j\not\subset P_i$ for each $1\leq i\leq N_1$ and $1\leq j\leq N_2$.

We denote the singular set of $\boldsymbol{C}$ in the embedded sense as $T(\boldsymbol{C})$, which is precisely defined as:

$$T(\boldsymbol{C}) := \{x\in \mathrm{spt}\|\boldsymbol{C}\| : \mathrm{spt}\|\boldsymbol{C}\| \text{ is not part of a hyperplane near } x\}.$$

We denote $T_\tau(\boldsymbol{C})$ as the τ-neighborhood of $T(\boldsymbol{C})$ for $\tau>0$.

Proposition 4.3.7

Let $n \geq 3$, and $\bar{n} < n-4+\frac{4}{n}$. Suppose M_j is a sequence of smooth, immersed, two-sided stable minimal hypersurfaces in $B_4^{n+1}(0)$ with $\dim(\operatorname{sing}(M_j)\cap B_4^{n+1}(0)) \leq \bar{n}$, such that M_j converge (as varifolds) to $\boldsymbol{C}\lfloor(B_4^{n+1}(0))$ as $j \to \infty$. Then,

$$\lim_{j\to\infty} \sup_{B_{\frac{1}{2}}^{n+1}(0)\cap M_j} |A_{M_j}| = 0.$$

In particular, $\boldsymbol{C}$ is a sum of hyperplanes with multiplicities, $\operatorname{sing} M_j \cap B_{\frac{1}{4}}^{n+1}(0) = \emptyset$, and $M_j \cap B_{\frac{1}{4}}^{n+1}(0)$ converges smoothly to $\boldsymbol{C}$ in $B_{\frac{1}{4}}^{n+1}(0)$ as immersions with q connected components, where $q = \Theta(\|\boldsymbol{C}\|, 0)$. ♠

Proof For each fixed $\tau > 0$, we know that $M_j \cap B_3^{n+1}(0)$ converges smoothly to $\boldsymbol{C}\lfloor(B_3^{n+1}(0)\backslash T_\tau(\boldsymbol{C}))$ as $j \to \infty$ by Proposition 4.3.5. For each $\tau > 0$, by the proof of Proposition 4.3.5, we know

$$\lim_{j\to\infty} \int_{B_2^{n+1}(0)\cap M_j \backslash T_\tau(\boldsymbol{C})} |A_{M_j}|^{2\alpha} = 0.$$

Now take $k = 0$, and let ϕ be a nonnegative cutoff function supported in $B_{\frac{3}{2}}^{n+1}(0)$, equal to 1 in $B_1^{n+1}(0)$, with $|\nabla\phi| \leq 4$. Applying Lemma 4.3.2 together with the Michael–Simon inequality [MS73], we have

$$\begin{aligned} \int_{M_j\cap B_1^{n+1}(0)} |A_{M_j}|^{\frac{2\alpha n}{n-2}} &\leq C\left(\int_{M_j\cap B_{\frac{3}{2}}^{n+1}(0)} |A_{M_j}|^{2\alpha}\right)^{\frac{n}{n-2}} \\ &\leq C\left(\int_{M_j\cap B_{\frac{3}{2}}^{n+1}(0)} |A_{M_j}|^{2}\right)^{\frac{n\alpha}{n-2}} \left(\mathcal{H}^n(M_j\cap B_{\frac{3}{2}}^{n+1}(0))\right)^{\frac{n(1-\alpha)}{n}}, \end{aligned} \tag{4.3.10}$$

for some $C = C(n)$. Note that the stability condition and Lemma 4.3.3 imply that the right-hand side of (4.3.10) is uniformly bounded. Hence,

$$\sup_{j>0} \int_{M_j\cap B_1^{n+1}(0)} |A_{M_j}|^{\frac{2\alpha n}{n-2}} < \infty.$$

By Hölder's inequality, we have

$$\begin{aligned} &\int_{M_j\cap T_\tau(\boldsymbol{C})\cap B_1^{n+1}(0)} |A_{M_j}|^{2\alpha} \\ \leq &\left(\int_{M_j\cap T_\tau(\boldsymbol{C})\cap B_1^{n+1}(0)} |A_{M_j}|^{\frac{2\alpha n}{n-2}}\right)^{\frac{n-2}{n}} \mathcal{H}^n(M_j\cap T_\tau(\boldsymbol{C})\cap B_1^{n+1}(0))^{\frac{2}{n}}. \end{aligned}$$

By a standard covering argument using the monotonicity formula, we know that

$$\mathcal{H}^n(M_j\cap T_\tau(\boldsymbol{C})\cap B_2^{n+1}(0)) \leq C\tau,$$

for some $C = C(n, \Theta(\|\boldsymbol{C}\|, 0))$ for j large enough. Hence, we have

$$\lim_{j\to\infty} \int_{B_1^{n+1}(0)\cap M_j} |A_{M_j}|^{2\alpha} = \lim_{j\to\infty} \int_{B_1^{n+1}(0)\cap M_j\cap T_\tau(\boldsymbol{C})} |A_{M_j}|^{2\alpha} \leq C\tau^{\frac{2}{n}}, \tag{4.3.11}$$

for some $C < \infty$ which is independent of τ. Since the left-hand side of (4.3.11) is independent of τ, and τ is arbitrary, we obtain

$$\lim_{j\to\infty} \int_{B_1^{n+1}(0)\cap M_j} |A_{M_j}|^{2\alpha} = 0.$$

Thus, we apply the ε-regularity theorem (Theorem 4.3.1) to conclude that

$$\lim_{j\to\infty}\sup_{B^{n+1}_{\frac{1}{2}}(0)\cap M_j}|A_{M_j}|=0,$$

which implies that each connected component of $M_j\cap B^{n+1}_{\frac{1}{2}}(0)$ converges to a hyperplane in the varifold sense. Furthermore, by Proposition 4.3.5, for j large enough, $\operatorname{sing}M_j\cap B^{n+1}_{\frac{1}{4}}(0)$ is empty, and $M_j\cap B^{n+1}_{\frac{1}{4}}(0)$ has exactly q connected components, each converging smoothly to a hyperplane in $B^{n+1}_{\frac{1}{4}}(0)$. ◆

We now prove Theorem 4.1.14. By Allard's compactness theorem (cf. Theorem 3.1.17), we obtain a stationary integral varifold V in $B^{n+1}_4(0)$ such that, up to a subsequence, $|M_j|$ converges to V in the varifold sense. Let $S=\operatorname{sing}\|V\|$ be the singular point set of V. We need to analyze the tangent cone $\boldsymbol{C}$ of V at $x_0\in S\cap B^{n+1}_{\frac{1}{2}}(0)$. Indeed, we have the following lemma.

Lemma 4.3.8

For any $\boldsymbol{C}\in\operatorname{VarTan}(V,x_0)$ for $x_0\in S\cap B^{n+1}_{\frac{1}{2}}(0)$, we can write $\boldsymbol{C}=\boldsymbol{C}'\times\mathbb{R}^{n-p}$ for some $p\geq 7$. ♠

Proof of Lemma 4.3.8 For any cone $\boldsymbol{C}$, we write $\mathcal{S}(\boldsymbol{C})$ (the spine of $\boldsymbol{C}$) to be the linear subspace containing all $x\in\mathbb{R}^{n+1}$ such that $\boldsymbol{C}$ is invariant under the translation along the line spanned by x. For any $x_0\in S$, we introduce the notion of iterated tangents of V at x_0 as follows. We say a collection of cones $\{\boldsymbol{C}_1,\boldsymbol{C}_2,\cdots,\boldsymbol{C}_N\}$ is iterated tangents of V at x_0 if $\boldsymbol{C}_1$ is the tangent cone of V at x_0, and $\boldsymbol{C}_{j+1}$ is the tangent cone of $\boldsymbol{C}_j$ at $x_j\in\operatorname{sing}\|\boldsymbol{C}_j\|\setminus\mathcal{S}(\boldsymbol{C}_j)$ for $1\leq j\leq N-1$. Moreover, we can choose iterated tangents satisfying the following properties:

1. Each $\boldsymbol{C}_j$ is not smoothly immersed (i.e., $\operatorname{sing}\|\boldsymbol{C}_j\|\neq\emptyset$).
2. $\dim(\mathcal{S}(\boldsymbol{C}_{j+1}))>\dim(\mathcal{S}(\boldsymbol{C}_j))$ for each $j=1,2,\cdots,N-1$.
3. $\boldsymbol{C}_N=\boldsymbol{C}'\times\mathbb{R}^{\dim(\mathcal{S}(\boldsymbol{C}_N))}$ where $\boldsymbol{C}'\setminus\{0\}$ is a smooth immersed cone after a suitable rotation in $\mathbb{R}^{n+1}$.
4. For each $1\leq j\leq N$, we can find a sequence of points $\{y_k\}$ with $y_k\to x_0$, a sequence of positive real numbers $\{r_k\}$ with $r_k\to 0^+$ as $k\to\infty$, such that $\eta_{y_k,r_k}(M_k)$ converges to $\boldsymbol{C}_j$ in the sense of varifolds and the convergence is smooth away from the singular set of $\boldsymbol{C}_j$ by Proposition 4.3.5 and Proposition 4.3.7.

In particular, the fourth condition implies that the smooth immersed part of $\boldsymbol{C}_j$ is stable, and the second condition implies N is a finite number.

The first three conditions are immediate from properties of tangent cones. The only nontrivial part is the last condition, which can be proved by induction on j. Suppose we have found y_k,r_k such that $\eta_{y_k,r_k}(M_k)$ converges to $\boldsymbol{C}_j$ in the sense of varifolds. Then, by the choice of $\boldsymbol{C}_{j+1}$, we know there exists ρ_k such that $\eta_{x_j,\rho_k}(\boldsymbol{C}_j)$ converges to $\boldsymbol{C}_{j+1}$ in the sense of varifolds. Thus, with $z_k=y_k+r_kx_j$ and $s_k=r_k\rho_k$, we have $\eta_{z_k,s_k}(M_k)$ converging to $\boldsymbol{C}_{j+1}$ in the varifold sense.

Now, let us determine the dimension of $\boldsymbol{C}'$. Note that $\boldsymbol{C}_N$ cannot be a hyperplane by the first condition.

If the dimension of $\boldsymbol{C}'$ is one, then $\boldsymbol{C}_N$ is the sum of distinct half-hyperplanes with multiplicity. But by Proposition 4.3.7, we know $\boldsymbol{C}_N$ can only be a sum of hyperplanes with multiplicity, which contradicts the first condition.

Therefore, we know $\boldsymbol{C}'$ has dimension at least two. But the fourth condition implies that $\boldsymbol{C}'$ is a smooth immersed stable cone away from $\{0\}$, and hence, $\boldsymbol{C}'$ has dimension at least 7.

Hence, by the second condition, we obtain $\dim(\boldsymbol{C})\geq n-7$ for any $\boldsymbol{C}\in\operatorname{VarTan}(V,x_0)$, and the lemma follows. ◆

Now, we are ready to finish the proof of Theorem 4.1.14.

Theorem 4.3.9

Let $\mathcal{V}$ be the collection of all the limit varifolds defined in Theorem 4.1.14. Then,

$$\dim(\operatorname{sing}(\|V\| \cap B_1)) \leq n - 7.$$

In particular, if $n = 7$, then $\operatorname{sing}(\|V\| \cap B_1)$ is discrete. ♡

Proof We denote $F^l = \{V \in \mathcal{V} : \mathcal{H}^l(\operatorname{sing} \cap B_1) > 0\}$.

Proposition 4.3.10

For each $V \in F^l$, there exists $\boldsymbol{C} \in \operatorname{VarTan}(V, x) \cap F^l$ for $\mathcal{H}^l-$a.e. $x \in \operatorname{sing}(\|V\|) \cap B_1$. ♠

Proof Recall that we actually have for $\mathcal{H}^l$-a.e. $x \in \operatorname{sing}(\|V\|) \cap B_1$, we have

$$\limsup_{r \to 0} \frac{\mathcal{H}^l_\infty(\operatorname{sing}\|V\| \cap B_r(x))}{\omega_n r^l} > 0.$$

We choose $r_i \to 0$ such that

$$\lim_{i \to \infty} \frac{\mathcal{H}^l_\infty(\operatorname{sing}\|V\| \cap B_{r_i}(x))}{\omega_n r_i^l} > 0.$$

By taking a subsequence, we can assume $(\eta_{x,r_i})_\# V$ converges to $\boldsymbol{C} \in \operatorname{VarTan}(V, x)$.

If $\mathcal{H}^l_\infty(\operatorname{sing}\|\boldsymbol{C}\|) = 0$, then for any $\varepsilon > 0$, we can find a covering of $\operatorname{sing}\|\boldsymbol{C}\|$ by balls $\{B_{s_j}(y_j)\}_{j=1}^\infty$ such that

$$\sum_{j=1}^\infty s_j^l < \varepsilon.$$

Note that $\operatorname{sing}\|\boldsymbol{C}\| \cap B_1(0)$ is compact, we know $\operatorname{sing}\|(\eta_{x,r_i})_\# V\| \cap B_1(0)$ can also be covered by $\{B_{s_j}(y_j)\}_{j=1}^\infty$ for i large enough. Thus, we have

$$\frac{\mathcal{H}^l_\infty(\operatorname{sing}\|V\| \cap B_{r_i}(x))}{r_i^l} < \varepsilon$$

for i large enough, which is a contradiction. ♦

Now we apply the above proposition iteratively to obtain a sequence of varifolds $\{V_k\}_{k=0}^K$ such that:

1. $V_0 = V$.
2. $V_{k+1} \in \operatorname{VarTan}(V_k, x_k)$ for some $x_k \in \operatorname{sing}(\|V_k\|) \cap B_1 \backslash \mathcal{S}(V_k)$.
3. $\dim(\mathcal{S}(V_{k+1})) > \dim(\mathcal{S}(V_k))$ for each $0 \leq k \leq K - 1$.
4. $\mathcal{H}^l(\operatorname{sing}(\|V_k\|) \cap B_1) > 0$ for each $0 \leq k \leq K$.
5. $V_K = \boldsymbol{C} \times \mathbb{R}^m$ where $\boldsymbol{C} \backslash \{0\}$ is a smooth immersed cone for some $m \geq 0$.

In particular, the last two conditions imply $m = l$. By Lemma 4.3.8, we know $l \leq n - 7$.

In the case $n = 7$, we have

$$\mathcal{H}^\alpha(\operatorname{sing}\|V_K\| \cap B_1) = 0$$

for any $\alpha > 0$ by the above result.

If $\operatorname{sing}(\|V\| \cap B_1)$ is not discrete, then we can find $x_j \in \operatorname{sing}(\|V\| \cap B_1)$ such that $x_j \to x_0 \in \operatorname{sing}(\|V\| \cap B_1)$. Now, up to a subsequence, we can assume $(\eta_{x_0, |x_j - x_0|})_\# V$ converges to $\boldsymbol{C} \in \operatorname{VarTan}(V, x_0)$ and we denote $\xi = \lim_{j \to \infty} \frac{x_j - x_0}{|x_j - x_0|} \neq 0$. So $\mathcal{S}(\boldsymbol{C})$ contains the line spanned by ξ. In particular, $\mathcal{H}^1(\operatorname{sing}(\|\boldsymbol{C}\|) \cap B_1) > 0$ which is a contradiction. Hence, $\operatorname{sing}(\|V\| \cap B_1)$ is discrete. ♦

4.4 Bellettini's Sheeting Theorem and Schoen–Simon Regularity

Theorem 4.4.1

Suppose M is a properly immersed, two-sided, stable minimal hypersurface in $B_4^{n+1}(0)$ with $\mathcal{H}^{n-2}(\mathrm{sing}(M)) < \infty$ and

$$\mathcal{H}^n(M \cap B_4^{n+1}(0)) \leq \Lambda.$$

Then there exists $\varepsilon = \varepsilon(n, \Lambda) > 0$ such that if

$$\int_{M \cap B_4^{n+1}(0)} dist(x, P)^2 d\mathcal{H}^n \leq \varepsilon$$

for $P = \{x_{n+1} = 0\}$, then $\pi : M \cap B_1^n(0) \times \mathbb{R} \backslash \pi^{-1}(\Sigma) \to B_1^n(0) \backslash \Sigma$ is a smooth projection map, where Σ is the projection of $sing(M)$ to P in $B_1^n(0)$.

In particular, if $\mathcal{H}^{n-2}(sing(M)) = 0$, then $\Sigma = \emptyset$ and $M \cap B_1^n(0) \times \mathbb{R}$ is a union of minimal graphs over $B_1^n(0)$. ♡

Theorem 4.4.2 (ε-regularity for the tilt)

Suppose M is a properly immersed, two-sided, stable minimal hypersurface in $B_4^{n+1}(0)$ with $\mathcal{H}^{n-2}(\mathrm{sing}(M)) < \infty$ and

$$\mathcal{H}^n(M \cap B_4^{n+1}(0)) \leq \Lambda.$$

Then, there exists $\varepsilon = \varepsilon(n, \Lambda) > 0$ such that if

$$\int_{M \cap B_4^{n+1}(0)} |x_{n+1}|^2 d\mathcal{H}^n \leq \varepsilon,$$

then

$$g(x) \leq C \left(\int_{M \cap B_2(0)} g^2(x) \right)^{\frac{4}{4+n}}.$$

Here, $g(x) = \sqrt{1 - (\nu \cdot e_{n+1})^2}$ where ν is the unit normal vector of M. ♡

Geometric meaning of g.

Definition 4.4.3 (Tilt excess)

*The **tilt excess** of M in $B_r^{n+1}(0)$ with respect to $P = \{x_{n+1} = 0\}$ is*

$$E_M(r, P) := r^{-n} \int_{M \cap B_r^{n+1}(0)} |T_x M - P|^2 d\mathcal{H}^n.$$

♣

For the codimensional one case, one can use $|\nu - e_{n+1}||\nu + e_{n+1}|$ as the distance between $T_x M$ and P. We have

$$|T_x M - P|^2 \simeq |\nu - e_{n+1}|^2 |\nu + e_{n+1}|^2 = 4(1 - (\nu \cdot e_{n+1})^2) = 4g^2.$$

This agrees with the standard definition of tilt excess up to a constant multiple.

Lemma 4.4.4

For any $k \in [0, \frac{1}{2n}]$, *we have*

$$\frac{1}{2n}\int_{\{g>k\}} |\nabla g|^2\left(1-\frac{k}{g}\right)\phi^2 \le \int_{\{g>k\}}(g-k)^2|\nabla\phi|^2.$$

for any Lipschitz function ϕ *supported in* $B^{n+1}_{\frac{3}{2}}(0)$. ♠

Proof Choose $(g-k)^+\phi$ as a test function in the stability inequality, together with

$$g\Delta g = -\frac{|\nabla g|^2}{1-g^2} + |A|^2(1-g^2)$$

and

$$\frac{|\nabla g|^2}{1-g^2} \le \frac{n-1}{n}|A|^2,$$

we can finish the proof for this lemma for the ϕ with compact support in the regular part of M. The general case can be obtained by a standard cut-off argument near the singular set of M.

The proof for ε-regularity theorem for the tilt is similar to the proof of the previous ε-regularity theorem for $|A|$, as we only need to repeat the de Giorgi iteration process.

The proof of the sheeting theorem is as follows.

Note that by the 1st variation formula, (by choosing $X = \phi^2 x_{n+1}e_{n+1}$) we have

$$\int g^2\phi^2 \le 4\int |x_{n+1}|^2|\nabla\phi|^2.$$

So

$$\int_{B_2} g^2 \le C\int_{B_4}|x_{n+1}|^2 \le C\varepsilon$$

Then, given any $\delta > 0$, we can choose ε small enough such that $g < \delta$ in $B^n_1 \times \mathbb{R} \cap \mathrm{reg}(M)$ by the ε-regularity theorem for the tilt. In particular, each small regular region of $M \cap B^n_1 \times \mathbb{R}$ can be written as a graph over a domain in P with small gradient. Then, by the connectedness of $M \cap B^n_1 \times \mathbb{R}$ away from the singular set, we know $M \cap B^n_1 \times \mathbb{R}\backslash\pi^{-1}(\Sigma)$ can be written as a graph of smooth multiple valued function (locally, it is the union of smooth graphs) over $B^n_1\backslash\Sigma$. This can be viewed as a covering map from $M \cap B^n_1 \times \mathbb{R}\backslash\pi^{-1}(\Sigma)$ to $B^n_1\backslash\Sigma$.

In addition, if $\mathcal{H}^{n-2}(\mathrm{sing}(M)) = 0$, then $B^n_1\backslash\Sigma$ is simply connected. So the covering map is trivial, and hence $M \cap B^n_1 \times \mathbb{R}$ is a union of minimal graphs over $B^n_1(0)\backslash\Sigma$. Again by the removable singularity theorem, each graph can be extended to a smooth minimal graph over $B^n_1(0)$. ♦

Theorem 4.4.5 (Schoen–Simon Regularity and Compactness, [SS81])

Let $\{M_k\}$ *be a sequence of embedded, stable, orientable minimal hypersurfaces in* $B^{n+1}_2(0)$ *with the following properties:*

1. $0 \in \bar{M}_k$ *for each* k.
2. $\mathcal{H}^{n-2}(\mathrm{sing} M_k) = 0$ *for each* k.
3. $\|M_k\|(B^{n+1}_2(0)) \le \Lambda$ *for some constant* $\Lambda > 0$ *independent of* k.

Then, up to a subsequence, M_k *converges in the varifold sense to a stable minimal hypersurface* M *in* $B^{n+1}_2(0)$, *which is smooth except for a closed singular set of Hausdorff dimension at most* $n-7$. ♡

Proof By Allard's compactness theorem, up to a subsequence we have $|M_i| \to V$ in the varifold sense.

Let $S = \mathrm{sing}\|V\|$ be the embedded singular point set of V. By Federer's dimension-reduction argument, it suffices to prove the following lemma.

Lemma 4.4.6

For any $\boldsymbol{C} \in \mathrm{VarTan}(V, x_0)$ *for* $x_0 \in S \cap B^{n+1}_{\frac{1}{2}}(0)$*, we can write* $\boldsymbol{C} = \boldsymbol{C}' \times \mathbb{R}^{n-p}$ *for some* $p \geq 7$*.* ♠

The proof of this lemma is similar to that of Lemma 4.3.8. We construct the iterated tangents of V at $x_0 \in S$ as $\{\boldsymbol{C}_1, \cdots, \boldsymbol{C}_N\}$ with the following:

1. $x_j \in \mathrm{sing}\|\boldsymbol{C}_j\| \backslash \mathcal{S}(\boldsymbol{C}_j)$ for each $1 \leq j \leq N-1$ and $\boldsymbol{C}_{j+1} \in \mathrm{VarTan}(\boldsymbol{C}_j, x_j)$.
2. Each $\boldsymbol{C}_j$ is not smoothly embedded (i.e., $\mathrm{sing}\|\boldsymbol{C}_j\| \neq \emptyset$).
3. $\dim(\mathcal{S}(\boldsymbol{C}_{j+1})) > \dim(\mathcal{S}(\boldsymbol{C}_j))$ for each $j = 1, 2, \cdots, N-1$.
4. $\boldsymbol{C}_N = \boldsymbol{C}' \times \mathbb{R}^{\dim(\mathcal{S}(\boldsymbol{C}_N))}$ where $\boldsymbol{C}' \backslash \{0\}$ is a smooth embedded cone after a suitable rotation in $\mathbb{R}^{n+1}$.
5. For each $1 \leq j \leq N$, we can find a sequence of points $\{y_k\}$ with $y_k \to x_0$, a sequence of positive real numbers $\{r_k\}$ with $r_k \to 0^+$ as $k \to \infty$, such that $\eta_{y_k, r_k}(M_k)$ converges to $\boldsymbol{C}_j$ in the sense of varifolds and the convergence is smooth away from the singular set of $\boldsymbol{C}_j$ by sheeting theorem.

We need to show that $\boldsymbol{C}'$ has dimension at least 7. If the dimension of $\boldsymbol{C}'$ is one, then $\boldsymbol{C}_N$ is the sum of distinct half-hyperplanes with multiplicity.

For simplicity, we denote

$$\boldsymbol{C}_N = \sum_{i=1}^{N_1} q_i \{(\cos\theta_i r, \sin\theta_i r, y) : r \geq 0, y \in \mathbb{R}^{n-1}\}.$$

We denote $\hat{M}_k = \eta_{y_k, r_k}(M_k)$. By the sheeting theorem, for any $\tau > 0$ and all large k, the set

$$\hat{M}_k \cap B_2^{n+1}(0) \backslash T_\tau(\boldsymbol{C}_N)$$

decomposes into $q = \sum_{i=1}^{N_1} q_i$ connected components. Each component is a smooth graph over the corresponding half-hyperplane in $\boldsymbol{C}_N$.

Now, for $\mathcal{H}^{n-1}$-a.e. point $y \in \mathbb{R}^{n-1} \cap B_1^{n-1}$, Sard's theorem and $\mathcal{H}^{n-2}(\mathrm{sing}\,\hat{M}_k) = 0$ imply that $\hat{M}_k \cap B_1^2 \times \{y\}$ consists of q embedded curves for large k.

Choose two pieces $N_1, N_2 \subset \hat{M}_k \backslash T_\tau(\boldsymbol{C}_N)$ such that N_i is a graph over a domain in $\{(\cos\theta_i r, \sin\theta_i r, y) : r \geq \tau, y \in \mathbb{R}^{n-1}\}$ and N_1, N_2 are connected by a curve γ as above.

Note that $|\nu(\gamma \cap \{r = \tau\} \cap N_1) - \nu(\gamma \cap \{r = \tau\} \cap N_2)| \geq \frac{1}{2}|\sin(\theta_1 - \theta_2)|$ for k large enough. So the integral

$$\int_{\gamma \cap \{r < \tau\}} |A| \geq \frac{1}{2}|\sin(\theta_1 - \theta_2)|.$$

Now, we integrate over all such curves γ for $\mathcal{H}^{n-1}$-a.e. point $y \in \mathbb{R}^{n-1} \cap B_1^{n-1}$, we have

$$\int_{\hat{M}_k \cap B_\tau^2 \times B_1^{n-1}} |A| \geq C \min_{i \neq j} |\theta_i - \theta_j|.$$

But on the other hand, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{\hat{M}_k \cap B_\tau^2 \times B_1^{n-1}} |A| &\leq \left(\mathcal{H}^n(\hat{M}_k \cap B_\tau^2 \times B_1^{n-1})\right)^{\frac{1}{2}} \left(\int_{\hat{M}_k \cap B_\tau^2 \times B_1^{n-1}} |A|^2\right)^{\frac{1}{2}} \\ &\leq C\sqrt{\tau}\end{aligned}$$

for some $C = C(n, \Lambda)$ independent of τ and k by the monotonicity formula and the stability inequality. Since τ is arbitrary, this is a contradiction.

Hence, we know $\boldsymbol{C}'$ has dimension at least 2. But the fifth condition implies that $\boldsymbol{C}'$ is a smooth embedded stable cone away from $\{0\}$, and hence $\boldsymbol{C}'$ has dimension at least 7. ♦

4.5 Sketch of Wickramasekera's Regularity Theorem

Theorem 4.5.1 (Wickramasekera's Regularity Theorem, [Wic14])

Suppose V_i is a sequence of stationary integral n-varifolds in $B_2^{n+1}(0)$ and V_i also satisfies the following conditions:

1. $0 \in \operatorname{spt}\|V_i\|$.
2. $\|V_i\|(B_2^{n+1}(0)) \leq \Lambda$ *for some constant* $\Lambda > 0$ *independent of* i.
3. *(Stability) Each V_i is stable in $B_2^{n+1}(0)$ on its regular set, i.e., for any $\phi \in C_c^1(\mathrm{reg}V_i)$,*
$$\int_{\mathrm{reg}V_i} |A_i|^2\phi^2 d\|V_i\| \leq \int_{\mathrm{reg}V_i} |\nabla\phi|^2 d\|V_i\|.$$
4. *(Alpha-Structural Hypothesis) There exists $\alpha \in (0,1)$ such that for each i, no point of $\operatorname{spt}\|V_i\| \cap B_1^{n+1}(0)$ has a neighborhood in which $\operatorname{spt}\|V_i\|$ is the union of three or more embedded $C^{1,\alpha}$ hypersurfaces-with-boundary meeting only along their common boundary.*

Then, up to a subsequence, V_i converges in the varifold sense to a stationary integral n-varifold V_∞ in $B_2^{n+1}(0)$, which is stable and whose singular set in $B_2^{n+1}(0)$ has Hausdorff dimension at most $n-7$. ♡

We write the class of such varifolds as $\mathcal{V}_\alpha(\Lambda)$.

Theorem 4.5.2 (Sheeting theorem)

Let $V_i \in \mathcal{V}_\alpha(\Lambda)$ such that $V_i \to q|B_2^n(0) \times \{0\}$ in the varifold sense. Then, for i large enough, we have
$$V_i\,|_{B_1^n \times \mathbb{R}} = \sum_{k=1}^{q} |graph u_{i,k}|,$$
with $u_{i,1} \leq \cdots \leq u_{i,q}$ and $u_{i,k}$ are smooth functions such that
$$\|u_{i,k}\|^2_{C^{1,\alpha}(B_1^n)} \leq C \int_{B_2} x_{n+1}^2 d\|V_i\|$$
for some constant $C = C(n,\alpha,\Lambda) > 0$ independent of i. ♡

Theorem 4.5.3 (Minimal Distance Theorem)

Suppose $\boldsymbol{C}$ is a classical cone. There is no sequence of varifolds $V_i \in \mathcal{V}_\alpha(\Lambda)$ such that $V_i \to \boldsymbol{C}|_{B_2(0)}$ in the varifold sense. ♡

Remark 4.5.4. Recall that a classical cone
$$\boldsymbol{C} := \sum_{i=1}^{N} q_i|H_i|$$
where H_i are n-dimensional half-hyperplanes in $\mathbb{R}^{n+1}$ such that they contains origin and share the same boundary and q_i are positive integers.

Sketch of the proof

1. The two theorems are proved simultaneously by induction on the density of planes and cones. Assume that the sheeting theorem holds for $q \leq q_0$ and that the minimal distance theorem holds for $\Theta(\boldsymbol{C},0) \leq q_0$.
2. Show that minimal distance theorem holds for $\Theta(\boldsymbol{C},0) \leq q_0+1$. It is enough to treat the densities $\Theta(\boldsymbol{C},0) = q_0 + \frac{1}{2}$ and q_0+1.
3. Show that the sheeting theorem holds for $q = q_0+1$.

4. Dimension reduction argument plus the classification of stable cones implies that the singular set of V_∞ has Hausdorff dimension at most $n-7$.

Proof of the minimal distance theorem

1. Assume the sheeting theorem holds for $q \leq q_0$ and the minimal distance theorem holds for $\Theta(\boldsymbol{C}, 0) \leq q_0$. Together with dimension reduction, this implies the desired regularity whenever $\Theta(\|V\|, X) \leq q_0$ for all $X \in B_2(0)$.
2. Suppose V is sufficiently close to a classical cone $\boldsymbol{C}$ in the varifold sense and $\Theta(\|V\|, 0) \geq \Theta(\|\boldsymbol{C}\|, 0)$. Then, outside neighborhood of the singular set of $\boldsymbol{C}$, using the sheeting theorem, we can write V as a union of smooth graphs over the half-hyperplanes in $\boldsymbol{C}$.
3. Establish the L^2-estimate for these graphs and key L^2 improvement in a smaller scale, following Simon's cylindrical-singularity argument [Sim93].
4. By iteration, we can obtain the set $\{\Theta(\|V\|, X) \geq \Theta(\|\boldsymbol{C}\|, 0)\}$ is $C^{1,\alpha}$-regular for some $\alpha > 0$, which implies V has classical singularity at 0, which is a contradiction.

Proof of the sheeting theorem

1. Same induction assumption as above.
2. Write V_i as a union of Lipschitz graphs $u_{i,k}$ over the plane $\{x_{n+1} = 0\}$ outside a small bad set.
3. Consider the blow-up class v, which is the limit of $E(V_i)^{-1} u_{i,k}$ where $E(V_i)$ is the L^2-excess of V_i.
4. $(\star)$ Prove that $v = (v_k)$ is harmonic.

First, we need a Lipschitz approximation for stationary varifolds.

Theorem 4.5.5 (Almgren's General Lipschitz Approximation for Stationary Integral Varifolds, [Alm00])

Fix $q \in \mathbb{Z}_{\geq 1}$ and $\sigma \in (0,1)$. Then there exists $\varepsilon_0 = \varepsilon_0(n, q, \sigma) \in (0,1)$ such that the following holds. Let V be a stationary integral n-varifold in $B_2^{n+1}(0)$ such that:

(i) $\dfrac{1}{\omega_n 2^n}\|V\|\big(B_2^{n+1}(0)\big) < q + \dfrac{1}{2}$ *and* $q - \dfrac{1}{2} \leq \dfrac{1}{\omega_n}\|V\|\big(\mathbb{R} \times B_1^n(0)\big) < q + \dfrac{1}{2}$;

(ii) $\hat{E}^2(V) := \displaystyle\int_{\mathbb{R}\times B_1^n(0)} |x^1|^2 \, \mathrm{d}\|V\|(X) < \varepsilon_0$.

Then there exists $\mathcal{H}^n$-measurable $\Sigma \subset B_\sigma^n(0)$ such that:

(a) $\mathcal{H}^n(\Sigma) + \|V\|(\mathbb{R} \times \Sigma) \leq C\, \hat{E}_V^2$;

(b) there exist Lipschitz functions $u_1, \ldots, u_q : B_\sigma^n(0) \to \mathbb{R}$ *with* $\mathrm{Lip}(u_j) \leq \frac{1}{2}$, $u_1 \leq \cdots \leq u_q$, $\displaystyle\sup_{B_\sigma^n(0)} |u_j| \leq C\, \hat{E}_V^{\frac{1}{n+1}}$, *and*

$$V \llcorner \big(\mathbb{R} \times (B_\sigma^n(0) \setminus \Sigma)\big) = \sum_{j=1}^{q} \big|\mathit{graph}(u_j|_{B_\sigma^n(0)\setminus\Sigma})\big|.$$

Here $C = C(n, q, \sigma) \in (0, \infty)$. ♡

Moreover, we can establish the following estimate

$$\int_{B_\sigma(0)} \sum_k |u_k|^2 + |Du_k|^2 \leq CE^2(V).$$

Definition 4.5.6 (Blow-up class)

We consider V_i as above such that $\hat{E}^2(V_i) \to 0^+$ and $\sigma_i \to 1$ as $i \to \infty$. The above estimate implies that

$\hat{u}_k := E^{-1}u_k$ satisfies

$$\sum_k \|u_k\|^2_{W^{1,2}(B_{\sigma_i}(0))} \le C.$$

Up to a subsequence, we can find a weak limit function v. The ***blow-up class*** *$\mathcal{B}_q$ is the collection of all such limits.* ♣

Theorem 4.5.7 (Properties of the blow-up class)

The blow-up class $\mathcal{B}_q$ has the following properties:

- ($\mathcal{B}1$) *$\mathcal{B}_q \subset W^{1,2}_{\mathrm{loc}}(B^n_1(0);\mathbb{R}^q) \cap L^2(B^n_1(0);\mathbb{R}^q)$.*
- ($\mathcal{B}2$) *If $v \in \mathcal{B}_q$, then $v^1 \le v^2 \le \cdots \le v^q$.*
- ($\mathcal{B}3$) *If $v \in \mathcal{B}_q$, then $\Delta v_a = 0$ in $B^n_1(0)$, where $v_a = q^{-1}\sum_{j=1}^q v^j$.*
- ($\mathcal{B}4$) *For each $v \in \mathcal{B}_q$ and each $z \in B^n_1(0)$, either ($\mathcal{B}4$ I) or ($\mathcal{B}4$ II) below holds:*
  - *($\mathcal{B}4$ I) (Hardt–Simon inequality [HS79]). For each $\rho \in \left(0, \frac{3}{8}(1-|z|)\right]$,*

    $$\sum_{j=1}^q \int_{B^n_{\rho/2}(z)} R_z^{2-n}\left(\frac{\partial}{\partial R_z}\left(\frac{v^j - v_a(z)}{R_z}\right)\right)^2 \mathrm{d}x \le C\rho^{-n-2}\int_{B^n_\rho(z)} \left|v-\ell_{v,z}\right|^2 \mathrm{d}x,$$

    where $R_z(x) = |x-z|$, $\ell_{v,z}(x) = v_a(z) + Dv_a(z)\cdot(x-z)$, and $v - \ell_{v,z} = \left(v^1 - \ell_{v,z}, \ldots, v^q - \ell_{v,z}\right)$.
  - *($\mathcal{B}4$ II) There exists $\sigma = \sigma(z) \in (0, 1-|z|]$ such that $\Delta v = 0$ in $B^n_\sigma(z)$.*
- ($\mathcal{B}5$) *If $v \in \mathcal{B}_q$, then:*
  - *($\mathcal{B}5$ I) $\tilde{v}_{z,\sigma}(\cdot) \equiv \|v(z+\sigma(\cdot))\|^{-1}_{L^2(B^n_1(0))}\, v(z+\sigma(\cdot)) \in \mathcal{B}_q$ for each $z \in B^n_1(0)$ and $\sigma \in \left(0, \frac{3}{8}(1-|z|)\right]$ whenever $v \not\equiv 0$ in $B^n_\sigma(z)$;*
  - *($\mathcal{B}5$ II) $v \circ \gamma \in \mathcal{B}_q$ for each orthogonal rotation γ of $\mathbb{R}^n$;*
  - *($\mathcal{B}5$ III) $\|v - \ell_v\|^{-1}_{L^2(B^n_1(0))}(v - \ell_v) \in \mathcal{B}_q$ whenever $v - \ell_v \not\equiv 0$ in $B^n_1(0)$, where $\ell_v(x) = v_a(0) + Dv_a(0)\cdot x$ for $x \in \mathbb{R}^n$ and $v - \ell_v = \left(v^1 - \ell_v, \ldots, v^q - \ell_v\right)$.*
- ($\mathcal{B}6$) *If $\{v_k\}_{k=1}^\infty \subset \mathcal{B}_q$, then there exists a subsequence $\{k'\}$ of $\{k\}$ and a function $v \in \mathcal{B}_q$ such that $v_{k'} \to v$ locally in $L^2(B^n_1(0))$ and locally weakly in $W^{1,2}(B^n_1(0))$.*
- ($\mathcal{B}7$) *If $v \in \mathcal{B}_q$ is such that for each $j = 1, \ldots, q$, there exist linear maps $L^j_1, L^j_2 : \mathbb{R}^n \to \mathbb{R}$ with $v^j(x^2, y) = L^j_1(x^2, y)$ if $x^2 > 0$, $v^j(x^2, y) = L^j_2(x^2, y)$ if $x^2 \le 0$, and $L^j_1(0, y) = L^k_2(0, y)$ for all $1 \le j, k \le q$ and $y \in \mathbb{R}^{n-1}$, where (x^2, y) are coordinates on $\mathbb{R}^n$, then $v^1 = v^2 = \cdots = v^q = L$ for some linear map $L : \mathbb{R}^n \to \mathbb{R}$.* ♡

Proof

- ($\mathcal{B}1$) This a consequence of the definition of the blow-up class.
- ($\mathcal{B}2$) This is based on $u_k \le u_{k+1}$.
- ($\mathcal{B}3$) This is from the stationary property of V_i.
- ($\mathcal{B}5$) We can consider a new sequence $V'_i = (\eta_{z,\sigma})_\# V_i$, where its blow-up limit is $\tilde{v}_{z,\sigma}$. Similarly, we can consider $V'_i = \gamma_\# V_i$, where its blow-up limit is $v \circ \gamma$. Finally, the last property is from the rotation involving direction e_{n+1}.
- ($\mathcal{B}6$) This follows from the compactness of the blow-up sequence.
- ($\mathcal{B}4$) This is one of the key properties of the blow-up class, and the place that stable condition is used. The idea is, either we have good density points accumulating at a given point, which implies the Hardt–Simon inequality. Or we have a density gap, which by our induction assumption, implies that v is harmonic.
- ($\mathcal{B}7$) This is the most technical part of the proof. It rules out the possibility that a blow-up class contains the

singular model of a classical cone.

◆

Theorem 4.5.8

For any $v \in \mathcal{B}_q$, we have v is smooth and harmonic. ♡

Proof This proof is similar to the classical dimensional reduction argument.

We define the "singular set" of v as the set of points $x \in B_1^n(0)$ such that $\mathcal{B}4\mathrm{I}$ holds.

First, we can study homogeneous degree-one functions in the blow-up class, namely tangent functions of a given function.

Claim. Each homogeneous 1 function in the blow-up class is a linear function.

Next, we can show that the "singular set" is empty by the dimensional reduction argument. ◆

After proving the blow-up class are all smooth and harmonic, we can show the sheeting theorem holds by using the standard regularity argument.

Proof of the property $\mathcal{B}7$

Suppose $v \in \mathcal{B}_q$ is such that for each $j = 1, \dots, q$, we can write

$$v^j(x_1, y) = \lambda^j x_1 \text{ if } x_1 > 0, \text{ and } v^j(x_1, y) = \mu^j x_1 \text{ if } x_1 \leq 0,$$

for some $\lambda^j, \mu^j \in \mathbb{R}$.

We define the cone $\boldsymbol{C}_k$ as the union of the graphs of the functions $\hat{E}_i \lambda^j x_1$ for $x_1 > 0$ and $\hat{E}_i \mu^j x_1$ for $x_1 \leq 0$.

We can define the fine excess $E(V_i)$ as follows:

$$E(V_i) := \int_{\mathbb{R} \times B_1^n(0)} d^2(X, \boldsymbol{C}_i) d\|V_i\|(X).$$

Clearly, we have $E(V_i) \leq \hat{E}(V_i)$. But we expect that $E(V_i)$ is much smaller than $\hat{E}(V_i)$.

Indeed, we can establish the following under suitable assumptions.

$$\lim \hat{E}^{-1}(V_i) E(V_i) = 0.$$

By the sheeting theorem, we can write V_i as a union of smooth graphs over the cones $\boldsymbol{C}_i$ for $|x_1| \geq \sigma$.

Now, we define h_i and w_i the vector valued function such that

$$V_i = \sum_{j=1}^{q} |h_i^j + \hat{E}_i \lambda^j x_1| + \sum_{j=1}^{q} |w_i^j + \hat{E}_i \mu^j x_1|,$$

and we expect that h_i^j and w_i^j much smaller than $\hat{E}_i$.

Establish the key L^2-estimate for h_i^j, w_i^j

Such estimate containst the following

$$\int_{B_{\frac{1}{2}}} \frac{|X^\perp|^2}{|X|^{n+2}} + \sum_{2 \leq j \leq n} |e_j^\perp|^2 + \frac{d^2(X, \operatorname{spt} \|C_j\|)}{|X|^{n+2-\mu}} \leq C \hat{E}_i^2.$$

We can also establish the uniform bound for $E^{-1}(V_i) h_i^j$ and $E^{-1}(V_i) w_i^j$ and obtain the limit function h and w. Such limit is called the fine blow-up limit.

Theorem 4.5.9

For the fine blow-up limit h and w, we know h, w are at least C^2 up to the boundary. ♡

Proof This relies on the key L^2 esimate using the fine excess.

First, we need to show the $C^{0,\alpha}$ estimate for h, w. We need to use the L^2-estimate to finish the proof.

Now, we need to show the $C^{2,\alpha}$ estimate for h, w. This is based on the stationary property of V_i and the definition of the blow-ups, and the properties of harmonic functions.

Note that each h, w are all harmonic in its interior, which is a consequence of the stationary property of V_i and definition of the blow-ups.

The key is actually the smoothness up to the boundary. ♦

Improvement of the fine excess at a smaller scale

Using the properties of the fine blow-up limit, we can improve the fine excess at a smaller scale as follows.

Under suitable assumptions, with $\hat{E}(V_i)$ small enough, $\hat{E}^{-1}(V_i)E(V_i)$ is small enough, we can find a new cone $\boldsymbol{C}_i'$, such that if we apply a rotation, a scaling to the original varifold V_i, we can obtain a new varifold V_i', such that the fine excess of V_i' with respect to $\boldsymbol{C}_i'$ is much smaller than the fine excess of V_i with respect to $\boldsymbol{C}_i$. One can think of this as follows. Under suitable assumptions, if one zooms in the original varifold V_i, it will look closer to a new cone $\boldsymbol{C}_i'$.

Iterative process

If such steps can be done infinitely many times for a given varifold V_i, we can obtain a limit cone $\boldsymbol{C}_{i,\infty}$. Note that the difference between $\boldsymbol{C}_{i,k}$ and $\boldsymbol{C}_{i,k+1}$ is much smaller than the difference between $\boldsymbol{C}_{i,k}$ and the plane $\{x_{n+1} = 0\}$, because the fine excess is much smaller than the original excess. Hence $\boldsymbol{C}_{i,\infty}$ is still a classical cone. These steps also imply that we can find varifolds $V_{i,k}$ which converge to $\boldsymbol{C}_{i,\infty}$ in the varifold sense.

This is a contradiction to the minimal distance theorem.

Chapter 5 Stable Bernstein Theorems

The classical Bernstein theorem asks when an entire minimal graph in $\mathbb{R}^{n+1}$ must be affine. From the variational point of view, an entire minimal graph is automatically stable, so a natural strengthening is to ask whether one can replace the graph assumption by stability alone. This leads to the stable Bernstein problem: classify complete stable minimal hypersurfaces in Euclidean space.

The statement below is the form relevant to the recent theory of stable immersions. The simply connected assumption rules out possible topological complications coming from immersions, while the two-sided assumption allows one to write the stability inequality with a globally defined normal field.

Conjecture 5.0.1 (Stable Bernstein conjecture)

Given $2 \leq n \leq 6$, suppose $M^n \to \mathbb{R}^{n+1}$ is a complete, two-sided, simply connected, stable minimal immersion. Then M is an affine hyperplane. ♡

Remark 5.0.2. Recall that Simons' cone

$$\mathbf{C} := \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

is a stable minimal hypersurface in $\mathbb{R}^8$. Thus the dimension range in the conjecture is sharp: the analogous statement is false for $n \geq 7$.

Here are some landmarks in the history of the problem.

1. In dimension $n = 2$, do Carmo–Peng [dCP79], Fischer–Colbrie–Schoen [FCS80], and Pogorelov [Pog81] proved the stable Bernstein theorem.
2. Schoen–Simon–Yau [SSY75] proved the result for $2 \leq n \leq 5$ under an area-growth hypothesis. A key step in their proof is an L^p curvature estimate of the form
$$\int |A_\Sigma|^{2p} u^{2p} d\Sigma \leq \int |\nabla u|^{2p} d\Sigma$$
for $2p < 4 + \sqrt{\frac{8}{n}}$.
3. Schoen–Simon [SS81] proved the corresponding classification for properly embedded stable minimal hypersurfaces with area growth.
4. Chodosh–Li [CL24] proved the case $n = 3$. Other approaches to this dimension were later given by Chodosh–Li [CL23] and by Catino–Mastrolia–Roncoroni [CMR24].
5. Chodosh–Li–Minter–Stryker [CLMS25] proved the case $n = 4$, and Mazet [Maz24] proved the case $n = 5$.
6. Cabré–Catino–Mari–Mastrolia–Roncoroni gave a Green-kernel proof of the $n = 3$ case and proved a sharp gradient estimate for Green kernels under spectral Ricci bounds [CCM+26].
7. The case $n = 6$ under an area-growth hypothesis follows from Bellettini [Bel25], together with the intrinsic–extrinsic area equivalence of Florit–Simon [FS26]; see Corollary 4.1.8. The unconditional $n = 6$ case is not addressed by these area-growth arguments and remains open. A main difficulty is that the present μ-bubble and spectral-Ricci volume-control methods do not yet seem strong enough to produce the needed area-growth input in this borderline dimension.

5.1 Stable Bernstein in Dimension Two

Theorem 5.1.1

Let M be a complete, stable minimal surface in $\mathbb{R}^3$. Then M is a plane. ♡

The proof presented here is based on the method of Fischer–Colbrie and Schoen [FCS80]. The argument has three main ingredients:

1. Properties of the Schrödinger operator $-\Delta+q$, or equivalently of the equation $(\Delta-q)g=0$, on complete Riemannian manifolds.
2. Properties of the operator $\Delta - aK$ on conformal metrics on the disc, where K is the Gauss curvature function and a is a constant.
3. Classification of the topology of stable minimal surfaces in 3-manifolds with non-negative scalar curvature.

5.1.1 Properties of differential operators on complete Riemannian manifolds

Let (M,g) be a complete n-dimensional Riemannian manifold, and let q be a smooth function on M. For any bounded domain $D \subset M$ with smooth boundary, we denote by $\lambda_1(D) < \lambda_2(D) \leq \lambda_3(D) \leq \cdots$ the Dirichlet eigenvalues of the Schrödinger operator $-\Delta + q$ on D, where Δ is the Laplace–Beltrami operator with respect to the metric g. Thus $\lambda_1(D)$ is the bottom of the quadratic form associated with the equation $(\Delta - q)g = 0$. The standard variational characterization of the first eigenvalue is given by

$$\lambda_1(D) = \inf\left\{\int_D \left(|\nabla f|_g^2 + qf^2\right) \mathrm{dvol}_g : \operatorname{spt} f \subset D, \int_D f^2 \mathrm{dvol}_g = 1\right\}, \tag{5.1.1}$$

where $|\nabla f|_g$ denotes the norm of the gradient of f with respect to the metric g, and dvol_g is the volume form induced by g. The following is a fundamental property:

Lemma 5.1.2

If D, D' are connected domains in M with $D \subset D'$, then $\lambda_1(D) \geq \lambda_1(D')$. Moreover, if $D' \setminus \bar{D} \neq \emptyset$, then $\lambda_1(D) > \lambda_1(D')$. ♠

We now state the main result of this section.

Theorem 5.1.3

The following conditions are equivalent:

(i) $\lambda_1(D) \geq 0$ *for every bounded domain* $D \subset M$*;*

(ii) $\lambda_1(D) > 0$ *for every bounded domain* $D \subset M$*;*

(iii) there exists a positive function g *satisfying the equation* $\Delta g - qg = 0$ *on* M*.* ♡

Proof (i) ⇒ (ii). This is a consequence of Lemma 5.1.2 since, for any bounded domain $D \subset M$ and any point $x_0 \in M$ we can choose R large enough so that the ball $B_R(x_0) = \{x \in M : \operatorname{dist}(x,x_0) < R\}$ satisfies $B_R(x_0) \setminus \bar{D} \neq \emptyset$ and we have $\lambda_1(B_R(x_0)) \geq 0$ by hypothesis.

(ii) ⇒ (iii). To prove the existence of a positive solution g of $\Delta g - qg = 0$ we fix a point $x_0 \in M$. For each $R > 0$ we consider the problem

$$\begin{cases} \Delta u - qu = 0 & \text{on } B_R(x_0), \\ u = 1 & \text{on } \partial B_R(x_0). \end{cases}$$

Since $\lambda_1(B_R(x_0)) > 0$, the Fredholm alternative thus implies the existence of the above problem.

We now prove that $u > 0$ on $B_R(x_0)$. It follows from the strong maximum principle that if $u \geq 0$ on $B_R(x_0)$, then $u > 0$ on $B_R(x_0)$. Suppose now that $\Omega = \{x \in B_R(x_0) : u(x) < 0\} \neq \emptyset$. Hence $\Omega \subset B_R(x_0)$ is a bounded domain and thus, by Lemma 5.1.2, $\lambda_1(\Omega) > 0$. Since $\Delta u - qu = 0$ on Ω and $u = 0$ on $\partial\Omega$, we would have $u \equiv 0$ in Ω contradicting the unique continuation property. We have shown that $u > 0$ on $B_R(x_0)$.

We now set $g_R(x) = u(x_0)^{-1}u(x)$ for $x \in M$. We have seen that g_R satisfies

$$\Delta g_R - qg_R = 0 \text{ on } B_R(x_0), \quad g_R(x_0) = 1,\ g_R > 0 \text{ on } B_R(x_0).$$

From the Harnack inequality, it follows that on any ball $B_\sigma(x_0)$, there is a constant C depending only on σ and M (independent of R) such that, for $R > 2\sigma$

$$g_R \leq C \text{ on } B_\sigma(x_0).$$

It now follows from standard elliptic theory that all derivatives of g_R are bounded uniformly (independent of R) on compact subsets of M. We may therefore choose a sequence $R_i \to \infty$ so that g_{R_i} converges along with its derivatives on any compact subset of M, and by taking a diagonal sequence we can arrange that g_{R_i} along with its derivatives, converges uniformly on compact subsets of M to a function g satisfying $\Delta g - qg = 0$ and $g(x_0) = 1$. Since g is not identically zero and $g \geq 0$ the strict maximum principle implies that $g > 0$. This finishes the proof that (ii) ⇒ (iii).

(iii) ⇒ (i). If $g > 0$ satisfies $\Delta g - qg = 0$ on M we define a new function $w = \log g$. We now calculate

$$\Delta w = q - |\nabla w|^2. \tag{5.1.2}$$

Let f be any function with compact support on M. Multiplying (5.1.2) by f^2 and integrating by parts, we obtain

$$-\int_M qf^2 \,\mathrm{dvol} + \int_M |\nabla w|^2 f^2 \,\mathrm{dvol} = 2\int_M f\langle \nabla f, \nabla w\rangle \,\mathrm{dvol}.$$

Applying the Schwarz inequality and the arithmetic-geometric mean inequality we have

$$2|f\langle \nabla f, \nabla w\rangle| \leq 2|f||\nabla f||\nabla w| \leq f^2|\nabla w|^2 + |\nabla f|^2.$$

Putting this into the above equation and canceling the terms $\int_M f^2|\nabla w|^2$ we obtain

$$-\int_M qf^2 \,\mathrm{dvol} \leq \int_M |\nabla f|^2 \,\mathrm{dvol}.$$

If D is any bounded domain and f is any function with support in D we have shown that

$$\int_D \left(|\nabla f|^2 + qf^2\right) \,\mathrm{dvol} \geq 0.$$

It now follows from (5.1.1) that $\lambda_1(D) \geq 0$. This finishes the proof of Theorem 5.1.3. ◆

The last part of the proof actually yields

Corollary 5.1.4

If $D \subset M$ is any bounded domain, and if there is a function $g > 0$ in D satisfying $\Delta g - qg = 0$, then $\lambda_1(D) \geq 0$. ♠

5.1.2 The Operator $\Delta - aK$ on Surfaces

Let M be the unit disc in the complex plane endowed with the metric $\mathrm{d}s^2 = \mu(z)|dz|^2$. We assume $\mathrm{d}s^2$ is a complete metric. Let K denote the Gaussian curvature of M and Δ the metric Laplacian, i.e., $\Delta f = \mu^{-1}(f_{xx} + f_{yy})$, where $z = x + iy$. The well-known formula for K is $K = -\frac{1}{2}\Delta \log \mu$. We shall prove the following theorem.

Theorem 5.1.5

Assume $\mathrm{d}s^2$ *is complete. For* $a > \frac{1}{2}$ *there is no positive solution* g *of* $\Delta g - aKg = 0$ *on* M*.* ♡

Remark 5.1.6. This is the key analytic input for determining the conformal type of stable minimal surfaces in $\mathbb{R}^3$. Fischer–Colbrie and Schoen [FCS80] prove the case $a = 1$, while the method of do Carmo and Peng [dCP79] gives the stated range $a > \frac{1}{2}$. For the Poincaré metric on the disc the critical value is $\frac{1}{4}$. Under the additional assumption $K \leq 0$, Kawai [Kaw88] proves the corresponding nonexistence result for $a > \frac{1}{4}$.

Proof We define a function h by $h = \mu^{-1/2}$. We see from the definition of K that $\Delta \log h = K$, i.e.,

$$\frac{\Delta h}{h} - \frac{|\nabla h|^2}{h^2} = K.$$

In particular, h satisfies

$$h\Delta h = Kh^2 + |\nabla h|^2.$$

Let $D \subset M$ be a bounded domain, and let ζ be a smooth function on M with compact support in D. We now calculate

$$\begin{aligned}
\int_M \left(|\nabla(\zeta h)|^2 + aK(\zeta h)^2\right) &= \int_M |\nabla\zeta|^2 h^2 + \frac{1}{2}\langle \nabla\zeta^2, \nabla h^2\rangle + \zeta^2|\nabla h|^2 + aK(\zeta h)^2 \\
&= \int_M |\nabla\zeta|^2 h^2 + \frac{1-a}{2}\langle \nabla\zeta^2, \nabla h^2\rangle + \frac{1-a}{2}\zeta^2|\nabla h|^2 \\
&\quad - \frac{a}{2}\zeta^2 h\Delta h + aK(\zeta h)^2 \,\mathrm{dvol} \\
&= \int_M \left(|\nabla\zeta|^2 h^2 + \frac{1-2a}{2}\zeta^2|\nabla h|^2\right) + \frac{1-a}{2}\langle \nabla\zeta^2, \nabla h^2\rangle \\
&\leq \int_M C|\nabla\zeta|^2 h^2 - \varepsilon \int_M |\nabla h|^2\zeta^2 \,\mathrm{dvol}.
\end{aligned}$$

So we have

$$\lambda_1(D)\int_M (\zeta h)^2 \,\mathrm{dvol} \leq \int_M |\nabla\zeta|^2 h^2 \,\mathrm{dvol} - \int_M |\nabla h|^2\zeta^2 \,\mathrm{dvol}. \tag{5.1.3}$$

Now define a smooth function $\zeta(r)$ for $r \in \mathbb{R}$ which satisfies

$$\begin{gathered}
\zeta(r) = 1 \text{ for } r \leq \frac{1}{2}R,\ \zeta(r) = 0 \text{ for } r \geq R, \\
\zeta \geq 0 \text{ for all } r, \quad |\zeta'| \leq \frac{3}{R} \text{ for all } r.
\end{gathered} \tag{5.1.4}$$

If r measures the metric distance to 0, and R is any positive number, then $\zeta(r)$ defines a Lipschitz function on M with support in $B_R(0)$. A standard approximation argument justifies this choice of ζ in (5.1.3). Then

$$\int_M |\nabla\zeta|^2 h^2 \,\mathrm{dvol} \leq \frac{9}{R^2}\int_M \mathrm{d}x\mathrm{d}y = \frac{9\pi}{R^2}.$$

Putting this into (5.1.3) we have

$$\lambda_1\left(B_R(0)\right)\int_M (\zeta h)^2 \,\mathrm{dvol} \leq \frac{9\pi}{R^2} - \int_M |\nabla h|^2\zeta^2 \,\mathrm{dvol}. \tag{5.1.5}$$

Since $\mu(z)|dz|^2$ is a complete metric on the disc, μ cannot be a constant function. Therefore, $|\nabla h|^2$ is not identically zero on M. Thus, by choosing R sufficiently large in (5.1.5), we conclude that $\lambda_1(B_R(0)) < 0$. By Theorem 5.1.3 this implies that there is no positive solution of $\Delta g - aKg = 0$ on M. This completes the proof of Theorem 5.1.5. ♦

The next result is an extension of Theorem 5.1.5 with an additional non-negative potential. It follows directly from Theorem 5.1.3, Theorem 5.1.5, and formula (5.1.1).

Corollary 5.1.7

Let $\mathrm{d}s^2 = \mu(z)|dz|^2$ *be a complete metric on the disc. If* $a \geq 1$ *and* P *is a non-negative function, then there is no positive solution* g *of* $\Delta g - aKg + Pg = 0$ *on* M.

5.1.3 Complete Stable Minimal Surfaces in 3-Manifolds

The stability of M is given by the following inequality:

$$\int_M \left[|\nabla f|^2 - \left(\operatorname{Ric}(e_3) + \sum_{i,j=1}^{2} h_{ij}^2 \right) f^2 \right] \mathrm{dvol} \geq 0, \tag{5.1.6}$$

where f is any function having compact support on M and $\operatorname{Ric}(e_3)$ is the Ricci curvature of N in the direction of e_3. We now do the rearrangement described in Schoen–Yau [SY79a]. The Gauss curvature equation says that

$$K = K_{12} + h_{11}h_{22} - h_{12}^2,$$

where K is the intrinsic Gaussian curvature of M and K_{ij} is the sectional curvature of N for the section determined by e_i and e_j.

Using minimality and symmetry of h_{ij} we have

$$K = K_{12} - \frac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2.$$

Inequality (5.1.6) may then be written in the form

$$\int_M \left[|\nabla f|^2 - \left(S - K + \frac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2 \right) f^2 \right] \mathrm{dvol} \geq 0, \tag{5.1.7}$$

where S is the scalar curvature of N given by $S = 2(K_{12} + K_{13} + K_{23})$.

Set

$$P := S - K + \frac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2.$$

According to (5.1.1), this inequality is equivalent to

$$\lambda_1(D; -\Delta - P) \geq 0$$

for every bounded domain $D \subset M$, i.e. to Theorem 5.1.3 with $q = -P$. The associated equation is given by the stability operator

$$\Delta + \left(S - K + \frac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2 \right). \tag{5.1.8}$$

We now classify the stable minimal surfaces in three-manifolds of non-negative scalar curvature.

Theorem 5.1.8

Let N *be a complete oriented 3-manifold of non-negative scalar curvature. Let* M *be an oriented complete stable minimal surface in* N. *There are two possibilities:*

(i) *If* M *is compact, then* M *is conformally equivalent to the sphere* S^2 *or* M *is a totally geodesic flat torus* T^2. *If* $S > 0$ *on* N, *then* M *is conformally equivalent to* S^2.

(ii) *If* M *is not compact, then* M *is conformally equivalent to the complex plane* $\mathbb{C}$ *or the cylinder* Λ. *If* M *is a cylinder and the absolute total curvature of* M *is finite, then* M *is flat and totally geodesic.*

If the scalar curvature of N is everywhere positive, then M cannot be a cylinder with finite total curvature.

If the Ricci curvature of N is non-negative, then the assumption of finite total curvature in (ii) can be removed. ♡

Before giving the proof of Theorem 5.1.8 we state the following corollary for the case when N is $\mathbb{R}^3$. This implies the classical Bernstein theorem [Ber17] for complete minimal graphs in $\mathbb{R}^3$.

Corollary 5.1.9

The only complete stable oriented minimal surface in $\mathbb{R}^3$ is the plane. ♠

Proof In this case the stability operator (5.1.8) becomes $\Delta - 2K$ and by Theorem 5.1.8 we know that M is conformally either $\mathbb{C}$ or Λ. By Theorem 5.1.3 there is a positive function g on M satisfying $\Delta g - 2Kg = 0$. If M is conformal to Λ we may lift g to the universal covering $\mathbb{C}$ of Λ. In either case we have a metric on $\mathbb{C}$ with $K \leq 0$ and a positive g satisfying $\Delta g - 2Kg = 0$. Thus $\Delta g \leq 0$, and g is a positive superharmonic function on $\mathbb{C}$ which must be constant. Therefore K is identically zero and hence $\sum_{i,j} h_{ij}^2 = -2K$ is identically zero. Consequently M is a plane. ♦

Remark 5.1.10. Observe that each of the four possibilities of Theorem 5.1.8 does occur. For example, $S^2 \times \mathbb{R}$ has positive scalar curvature and has a stable S^2, $T^2 \times \mathbb{R}$ is flat and has a stable T^2. We can choose a metric on $\mathbb{C}$ of positive Gaussian curvature and by crossing with $\mathbb{R}$ construct a metric of positive scalar curvature on $\mathbb{R}^3$ having a stable $\mathbb{C}$. Similarly $\Lambda \times \mathbb{R}$ has a flat metric with a stable Λ.

Proof of Theorem 5.1.8 Case (i) was observed by Schoen–Yau [SY79a] and follows by choosing f identically equal to one in inequality (5.1.7) to obtain

$$\int_M \left(S + \frac{1}{2} \sum h_{ij}^2 \right) \text{dvol} \leq \int_M K \, \text{dvol}.$$

The Gauss-Bonnet theorem now implies that M is the sphere or the torus. In the torus case $S \equiv 0$ on M. The stability operator reduces to $\Delta - K$ and its first eigenvalue is

$$\lambda_1 \equiv \inf \left\{ \int_M \left(|\nabla f|^2 + K f^2 \right) \text{dvol} : \int_M f^2 = 1 \right\}.$$

Since $\lambda_1 \geq 0$ by stability and $\int_M K \, \text{dvol} = 0$ we conclude that $\lambda_1 = 0$ and the constant function $f \equiv 1$ satisfies $\Delta f - K f = 0$ which implies that $K \equiv 0$.

To prove case (ii), we first show that the universal covering of M is conformally equivalent to $\mathbb{C}$. If this is not true, then M is covered by the disc. Using stability and Theorem 5.1.3 we have a positive function g on M satisfying

$$\Delta g - Kg + \left(S + \frac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2 \right) g = 0.$$

Lifting g to the disc we obtain a positive solution of this equation on the disc endowed with a complete metric. Since $S + \frac{1}{2} \sum_{i,j=1}^{2} h_{ij}^2 \geq 0$, this yields a contradiction by Corollary 5.1.7. Thus we have shown that M is conformally covered by $\mathbb{C}$ and hence M is either conformally equivalent to $\mathbb{C}$ or M is conformal to a cylinder Λ.

If M is a cylinder, let $z = x + iy$ be a complex coordinate for M so that $|dz|^2$ is the flat metric on M, and the given metric on M is $\mathrm{d}s^2 = \mu(z)|dz|^2$. Fix a point $z_0 \in M$ and let r be the distance from z_0 taken with respect to the flat metric. For any $R > 0$, choose $\zeta(r)$ satisfying (5.1.4). Substituting ζ for f in (5.1.7) and using (5.1.4) and the conformal invariance of the Dirichlet integral we have

$$\frac{9}{R^2}\int_{B_R(z_0)} \mathrm{d}x\mathrm{d}y - \int_M \left(S - K + \frac{1}{2}\sum h_{ij}^2\right) f^2 \,\mathrm{dvol} \geq 0,$$

where $B_R(z_0)$ is the ball taken with respect to the flat metric. Since $\int_{B_R(z_0)} \mathrm{d}x\mathrm{d}y$ has growth bounded by a constant times R and we are assuming $\int_M |K| \,\mathrm{dvol} < \infty$ we can use the dominated convergence theorem to let R tend to infinity to achieve

$$\int_M \left(S + \frac{1}{2}\sum_{i,j=1}^{2} h_{ij}^2\right) \mathrm{dvol} \leq \int_M K \,\mathrm{dvol}.$$

Recall that the Cohn–Vossen inequality states that if M is a complete non-compact surface with finite total curvature, then $\int_M K \,\mathrm{dvol} \leq 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. Since M is topologically a cylinder, we have $\chi(M) = 0$. Thus the Cohn–Vossen inequality gives $\int_M K \,\mathrm{dvol} \leq 0$. Since $S + \frac{1}{2}\sum h_{ij}^2 \geq 0$ we conclude that M is totally geodesic and $S \equiv 0$ on M.

Hence the stability operator reduces to $\Delta - K$. By Theorem 5.1.3 there is a positive function g on M satisfying $\Delta g - Kg = 0$. Set $w = \log g$. Computing we have

$$\Delta w = K - |\nabla w|^2.$$

Choosing ζ as above, we multiply by ζ^2 and integrate by parts to get

$$\int_M |\nabla w|^2 \zeta^2 \,\mathrm{dvol} = \int_M \zeta^2 K \,\mathrm{dvol} + 2\int_M \zeta\langle\nabla\zeta, \nabla w\rangle \leq \int_M \zeta^2 K + \frac{1}{4}|\nabla w|^2\zeta^2 \,\mathrm{dvol} + 4|\nabla\zeta|^2.$$

The Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality give

$$2|\zeta||\langle\nabla\zeta, \nabla w\rangle| \leq \frac{1}{4}\zeta^2|\nabla w|^2 + 4|\nabla\zeta|^2.$$

Therefore,

$$\frac{3}{4}\int_M |\nabla w|^2\zeta^2 \,\mathrm{dvol} \leq \int_M \zeta^2 K \,\mathrm{dvol} + 4\int_M |\nabla\zeta|^2 \,\mathrm{dvol}.$$

Letting $R \to \infty$ as above, we conclude that

$$\frac{3}{4}\int_M |\nabla w|^2 \,\mathrm{dvol} \leq \int_M K \,\mathrm{dvol}.$$

Thus w is constant, so g is constant and we have $K \equiv 0$.

In case N has non-negative Ricci curvature, we write the stability operator as $\Delta + \mathrm{Ric}(e_3) + \sum_{i,j=1}^{2} h_{ij}^2$ and note that the proof that M is totally geodesic now follows as in the previous paragraph (without the assumption of finite total curvature). From the previous proof we also get that

$$\mathrm{Ric}(e_3) = K_{13} + K_{23} = 0 \text{ on } M.$$

Since

$$\mathrm{Ric}(e_1) = K_{12} + K_{13} \geq 0, \quad \mathrm{Ric}(e_2) = K_{12} + K_{23} \geq 0,$$

we have

$$\mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) = 2K_{12} = 2K \geq 0.$$

Thus the Gauss curvature of M is non-negative and, since M is a cylinder, we have $K \equiv 0$. This completes the proof of Theorem 5.1.8. ◆

5.2 Higher-Dimensional Stable Bernstein Theorems

We outline the strategy for the unconditional stable Bernstein theorem in dimensions $n = 3, 4, 5$. The argument has three main steps: a conformal change producing positive *spectral* bi-Ricci curvature, the construction of μ-bubbles, and uniform area bounds for these bubbles via spectral Ricci estimates. The conformal and spectral curvature estimates are proved in §5.2.1; the μ-bubble reduction is treated in §5.2.2, and the spectral Ricci estimates are recorded in §5.2.6.

Step 1: Conformal change and spectral bi-Ricci curvature. Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be a complete, two-sided, stable minimal immersion. Fix $0 \in M$ and write $r(x) = |x|$ for the Euclidean distance from the origin. Following Schoen–Simon–Yau and the recent stable Bernstein literature, we pass to the conformal metric

$$\tilde{g} = r^{-2} g, \qquad g = \varphi^* \delta,$$

where g is the induced metric. Stability of M yields a spectral inequality for the conformal Laplacian with a curvature potential. To state it uniformly in dimension, we introduce the *bi-Ricci curvature*.

Definition 5.2.1 (Bi-Ricci curvature)

Let (N, g) be a Riemannian manifold of dimension $m \geq 2$. For orthonormal $v, w \in T_pN$, define

$$\operatorname{BiRic}(v, w) := \operatorname{Ric}(v, v) + \operatorname{Ric}(w, w) - R(v, w, v, w).$$

As functions on N, we write BiRic *and* Ric *for the minimum over unit directions (with $v \perp w$ for* BiRic*):*

$$\operatorname{BiRic}(p) := \inf_{\substack{v, w \in T_pN \\ |v|=|w|=1,\ \langle v, w\rangle = 0}} \operatorname{BiRic}(v, w), \qquad \operatorname{Ric}(p) := \inf_{\substack{v \in T_pN \\ |v|=1}} \operatorname{Ric}(v, v).$$

♣

In dimension $m = 3$, the bi-Ricci curvature is independent of the chosen orthonormal pair:

$$\operatorname{BiRic}(v, w) = \frac{1}{2} R.$$

This is why the three-dimensional part of the argument is often stated as a spectral scalar-curvature condition.

We use *spectral curvature condition* to mean a lower bound for the first eigenvalue of a Schrödinger operator whose potential is built from one of these curvature quantities. The coefficient depends on the dimension and on the normalization used in the μ-bubble argument.

In the conformal metric $\tilde{g} = r^{-2} g$, stability gives the concrete inputs needed below: for $n = 3$,

$$\lambda_1 \left(-\tilde{\Delta} + \frac{1}{2} \tilde{R} \right) \geq \frac{3}{2}$$

by the conformal scalar-curvature computation in §5.2.1, while for $n = 4$,

$$\lambda_1 \left(-\tilde{\Delta} - \widetilde{\operatorname{BiRic}} \right) \geq 1$$

after the corresponding conformal bi-Ricci normalization.

Thus, after conformal reparametrization, the problem is reduced to studying a manifold with non-negative *spectral* bi-Ricci curvature (in dimension three this is precisely non-negative spectral scalar curvature). This replaces the extrinsic curvature input of the minimal hypersurface by an intrinsic spectral curvature hypothesis on $(M, \tilde{g})$.

Step 2: Construction of μ-bubbles. The second step is to produce separating hypersurfaces inside a collar of $(M, \tilde{g})$ with controlled geometry. These are μ-bubbles.

Classically, a *soap bubble* is a hypersurface of constant mean curvature (CMC) enclosing a prescribed volume—the archetypal prescribed mean curvature problem. More generally, one studies hypersurfaces whose mean curvature is a prescribed function of position and geometry; Gromov introduced and systematically used such constructions in his work on scalar curvature, naming them μ*-bubbles* (the name reflects the measure μ specifying the prescription). In the present setting, one considers a Riemannian manifold (N, g) with boundary $\partial N = \partial_+ N \cup \partial_- N$ and seeks a domain $\Omega \subset N$ whose boundary $\Sigma = \partial\Omega \setminus \partial_- N$ (the μ-bubble) solves a variational problem of prescribed mean curvature type: schematically, one minimizes a functional of the form

$$\mathcal{A}(\Omega) = \int_\Sigma w \, d\mathcal{H}^{m-1} - \int_\Omega w \, h \, d\mathcal{H}^m$$

for a weight w (often the first eigenfunction of a suitable operator on N) and a carefully chosen function h determined by the curvature of N. Under the spectral bi-Ricci condition from Step 1, such a minimizer exists and provides a μ-bubble Σ separating $\partial_- N$ from $\partial_+ N$ (Theorem 5.2.8).

In recent years, μ-bubbles have become a central tool in positive scalar curvature geometry and have led to breakthroughs in the classification of manifolds with positive scalar curvature, the resolution of the Riemannian positive mass theorem in many settings, and the stable Bernstein program in dimensions three through five. The method is flexible: by choosing the prescription (the μ-data and the function h), one encodes the desired curvature inequality into the Euler–Lagrange equation of the μ-bubble.

Step 3: Volume and diameter estimates for μ-bubbles. The third step controls the size of each μ-bubble Σ. This is analogous in spirit to the *dimension reduction* in the Schoen–Yau proof of the positive mass theorem: one does not work directly on the full ambient manifold, but on a hypersurface of one dimension lower whose geometry is more rigid.

After Step 2, each μ-bubble Σ is a closed hypersurface in N (of dimension $m-1$) lying in a thin collar between $\partial_- N$ and $\partial_+ N$. Using the construction and the spectral bi-Ricci hypothesis on N, one shows that Σ itself carries a *spectral non-negative Ricci* condition: the first eigenvalue of an operator of the form $-\Delta^\Sigma + \alpha \operatorname{Ric}$ is non-negative for some $\alpha \in (0, 2)$ depending on n. One then studies manifolds with spectral non-negative Ricci curvature by methods similar to Bray's proof of the Bishop–Gromov comparison: spectral bounds on $-\Delta + \operatorname{Ric}$ imply diameter and volume upper bounds (Theorem 5.2.14 and the discussion in §5.2.6). In particular, each component of $\partial\Omega$ has area and intrinsic diameter bounded by constants depending only on the spectral bound, not on the size of the collar.

Conclusion of the proof. Combining Steps 1–3, one obtains separating μ-bubbles in long conformal annuli with uniform $\tilde{g}$-area and diameter bounds. As explained in the proof of the Euclidean volume-growth estimate in §5.2.6, these conformal estimates imply

$$|B^M_\rho(0)|_g \le C\rho^n$$

for every $\rho > 0$. Thus the μ-bubble argument supplies the area-growth hypothesis needed for the stable Bernstein theorems with area growth, and the cited classification results force M to be an affine hyperplane. This completes the proof sketch; the conformal and spectral bi-Ricci estimates used in Steps 1–3 are proved in §5.2.1.

5.2.1 Conformal change of the metric on minimal hypersurfaces

Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be a complete, two-sided, stable minimal immersion with induced metric $g = \varphi^*\delta$. Fix $0 \in M$ and write $r(x) = |x|$. Throughout we use the conformal metric

$$\tilde{g} = r^{-2} g, \qquad d\tilde{\mu} = r^{-n} \, d\mu, \qquad \tilde{\nabla} = r^2 \nabla^M, \quad e_i = r\tilde{e}_i.$$

We use the bi-Ricci convention fixed above: BiRic denotes the minimum over orthonormal two-frames, and in dimension three $\operatorname{BiRic} = \frac{1}{2} R$.

Theorem 5.2.2

Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be a complete, two-sided, stable minimal immersion with induced metric $g = \varphi^*\delta$. Then, the conformal metric $\tilde{g} = r^{-2}g$ satisfies

$$-\tilde{\Delta} + \frac{2}{n-2}\widetilde{\mathrm{BiRic}} \geq \frac{2(-n^3+6n^2-4n-8)}{8(n-2)}$$

where $\tilde{\Delta}$ is the conformal Laplacian and $\widetilde{\mathrm{BiRic}}$ is the conformal bi-Ricci curvature. ♡

Under a general conformal change $\tilde{g} = f^{-2}g$, we first relate the intrinsic Hessian on M to the ambient Euclidean Hessian.

Lemma 5.2.3 (Hessian on a hypersurface)

Let $M^n \subset \mathbb{R}^{n+1}$ be a hypersurface with unit normal ν, second fundamental form $A(X,Y) = \langle D_X Y, \nu\rangle$, and ambient connection D. For $f \in C^\infty(M)$ and tangent vector fields X, Y on M,

$$\mathrm{Hess}^M f(X,Y) := \langle \nabla^M_X \nabla^M f, Y\rangle = D^2 f(X,Y) + A(X,Y)\langle Df, \nu\rangle,$$

where $D^2 f(X,Y) := D_X(D_Y f)$ and ∇^M is the Levi-Civita connection of the induced metric. ♠

Proof Write the ambient gradient $Df = \nabla^M f + \langle Df, \nu\rangle\,\nu$, so $\nabla^M f$ is the tangential part. By direct computation,

$$\begin{aligned}\nabla^M_X \nabla^M_Y f &= XY(f) - \nabla^M_X Y f = XY(f) - D_X Y(f) + \langle D_X Y, \nu\rangle\langle Df, \nu\rangle \\ &= D^2 f(X,Y) + A(X,Y)\langle Df, \nu\rangle.\end{aligned}$$

♦

For the conformal metric $\tilde{g} = e^{2\varphi}g$, the change of Riemannian curvatures is given by

$$\tilde{R}_{ijij} = e^{2\varphi}R_{ijij} - e^{2\varphi}(T_{ii} + T_{jj})$$

where

$$T_{ij} = \nabla_i\nabla_j\varphi - \nabla_i\varphi\nabla_j\varphi + \frac{1}{2}|\nabla\varphi|^2 g_{ij}.$$

For $\varphi = -\log r$, we have

$$T_{ii} = -\nabla^2_{ii}\log r - |r_i|^2/r^2 + \frac{1}{2}|\nabla r|^2/r^2 = -D^2_{ii}\log r - \frac{|r_i|^2}{r^2} + \frac{1}{2}|\nabla r|^2/r^2 - A_{ii}\langle Dr, \nu\rangle/r$$

So

$$T_{ii} = -\frac{1}{r^2} + |r_i|^2/r^2 + \frac{1}{2}|\nabla r|^2/r^2 - A_{ii}\langle Dr, \nu\rangle/r$$

For the conformal metric $\tilde{g} = r^{-2}g$ with $r(x) = |x|$, $\varphi = -\log r$, the conformal curvatures satisfy

$$\begin{aligned}\tilde{R}_{\widetilde{ijij}} &= r^4\tilde{R}_{ijij} \\ &= r^2 R_{ijij} - r^2(T_{ii} + T_{jj}) \\ &= r^2 R_{ijij} - (-2 + |r_i|^2 + |r_j|^2 + |\nabla r|^2 - A_{ii}\langle rDr, \nu\rangle - A_{jj}\langle rDr, \nu\rangle) \\ \widetilde{\mathrm{BiRic}}(\tilde{e}_1, \tilde{e}_2) &= r^2\mathrm{BiRic}(e_1, e_2) + 2(n-3) - (2n-1)|\nabla r|^2 - (n-3)(|r_i|^2 + |r_j|^2) \\ &\quad + (n-3)(A_{11} + A_{22})\langle rDr, \nu\rangle,\end{aligned}$$

We compute BiRic on M using the Gauss equation.

Proposition 5.2.4

Let $\{e_1, \ldots, e_n\}$ *be a local orthonormal frame on a minimal hypersurface* $M^n \subset \mathbb{R}^{n+1}$. *Then*

$$\mathrm{BiRic}(e_1, e_2) = -\sum_{i=1}^{n} A_{1i}^2 - \sum_{j=2}^{n} A_{2j}^2 - A_{11}A_{22}.$$

♠

Proof Using the Gauss equation, we compute

$$\begin{aligned}\mathrm{BiRic}(e_1, e_2) &= \sum_{i=2}^{n} R_{1i1i} + \sum_{j=3}^{n} R_{2j2j} \\ &= \sum_{i=2}^{n}(A_{11}A_{ii} - A_{1i}^2) + \sum_{j=3}^{n}(A_{22}A_{jj} - A_{2j}^2) \\ &= -\sum_{i=1}^{n} A_{1i}^2 - \sum_{j=2}^{n} A_{2j}^2 - A_{11}A_{22},\end{aligned}$$

where the last line follows since $\operatorname{tr} A = 0$ for a minimal hypersurface.

♦

Now, we choose $\tilde{e}_1$ and $\tilde{e}_2$ such that $\widetilde{\mathrm{BiRic}}(\tilde{e}_1, \tilde{e}_2)$ takes the minimum value. Then, we have

$$\begin{aligned}\widetilde{\mathrm{BiRic}} = &-\sum_{i=1}^{n} A_{1i}^2 - \sum_{j=2}^{n} A_{2j}^2 - A_{11}A_{22} + 2(n-3) - (2n-1)|\nabla r|^2 - (n-3)(|r_i|^2 + |r_j|^2) \\ &+ (n-3)(A_{11} + A_{22})\langle rDr, \nu\rangle.\end{aligned}$$

Proposition 5.2.5

For $n \geq 3$, *we have*

$$r^2|A|^2 \geq \frac{2}{n-2}\left((3n-3) - (2n-1)|dr|^2 - \widetilde{\mathrm{BiRic}}\right).$$

♠

Proof Recall that

$$\begin{aligned}&r^2\left(\sum_{i=1}^{n} A_{1i}^2 + \sum_{j=2}^{n} A_{2j}^2 + A_{11}A_{22}\right) + (n-3)\langle \vec{x}, \nu\rangle(A_{11} + A_{22}) \\ &\quad = (4n-6) - (2n-1)|dr|^2 - (n-3)\big(dr(e_1)^2 + dr(e_2)^2\big) - \widetilde{\mathrm{BiRic}}.\end{aligned} \tag{5.2.1}$$

Since $\langle \vec{x}, \nu\rangle = r\, dr(\nu)$, we use Young's inequality to obtain

$$|(n-3)\langle \vec{x}, \nu\rangle(A_{11} + A_{22})| \leq (n-3)dr(\nu)^2 + \frac{n-3}{4}r^2(A_{11} + A_{22})^2.$$

Combined with (5.2.1) and the fact that

$$dr(e_1)^2 + dr(e_2)^2 + dr(\nu)^2 \leq 1,$$

we have

$$\begin{aligned}&r^2\left(\sum_{i=1}^{n} A_{1i}^2 + \sum_{j=2}^{n} A_{2j}^2 + A_{11}A_{22} + \frac{n-3}{4}(A_{11} + A_{22})^2\right) \\ &\quad \geq (3n-3) - (2n-1)|dr|^2 - \widetilde{\mathrm{BiRic}}.\end{aligned}$$

Now we compute, using the fact that $\operatorname{Tr} A = 0$,

$$\begin{aligned} & A_{11}^2 + A_{22}^2 + A_{11}A_{22} + \frac{n-3}{4}(A_{11}+A_{22})^2 \\ &= \frac{1}{2}(A_{11}^2 + A_{22}^2) + \frac{n-1}{4}(A_{11}+A_{22})^2 \\ &= \frac{1}{2}(A_{11}^2 + A_{22}^2) + \frac{n-1}{4}\sigma(A_{11}+A_{22})^2 + \frac{n-1}{4}(1-\sigma)(A_{33}+\cdots+A_{nn})^2 \\ &\le \left(\frac{1}{2} + \frac{n-1}{2}\sigma\right)(A_{11}^2 + A_{22}^2) + \frac{(n-1)(n-2)}{4}(1-\sigma)(A_{33}^2+\cdots+A_{nn}^2) \\ &= \frac{n-2}{2}(A_{11}^2+\cdots+A_{nn}^2), \end{aligned}$$

where we took

$$\sigma = \frac{n-3}{n-1}$$

in the last line. Hence, for $n \geq 3$, we have

$$\begin{aligned} \frac{n-2}{2} r^2 |A|^2 &\geq r^2 \left(\frac{n-2}{2} \sum_{i=1}^{n} A_{ii}^2 + \sum_{i=2}^{n} A_{1i}^2 + \sum_{j=3}^{n} A_{2j}^2 \right) \\ &\geq r^2 \left(\sum_{i=1}^{n} A_{1i}^2 + \sum_{j=2}^{n} A_{2j}^2 + A_{11}A_{22} + \frac{n-3}{4}(A_{11}+A_{22})^2 \right) \\ &\geq (3n-3) - (2n-1)|dr|^2 - \widetilde{\mathrm{BiRic}}. \end{aligned}$$

Therefore,

$$r^2|A|^2 \geq \frac{2}{n-2}\left((3n-3) - (2n-1)|dr|^2 - \widetilde{\mathrm{BiRic}}\right),$$

and the proposition follows. ♦

Proposition 5.2.6

For any $\psi \in C_c^{0,1}(N, \tilde{g})$, we have

$$\int_N |\tilde{\nabla}\psi|_{\tilde{g}}^2 \, d\tilde{\mu} \geq \int_N \left(r^2|A|^2 - \frac{n(n-2)}{2} + \left(\frac{n(n-2)}{2} - \frac{(n-2)^2}{4} \right) |dr|^2 \right) \psi^2 \, d\tilde{\mu}.$$

♠

Proof Recall that in the conformal metric, we have

$$d\tilde{\mu} = r^{-n} d\mu \qquad \text{and} \qquad |\tilde{\nabla} f|_{\tilde{g}}^2 = r^2 |\nabla f|^2.$$

Then the stability inequality for M implies

$$\int_N r^{n-2} |\tilde{\nabla} f|_{\tilde{g}}^2 \, d\tilde{\mu} \geq \int_N r^{n-2}(r^2|A|^2) f^2 \, d\tilde{\mu}$$

for any $f \in C_c^{0,1}(N, \tilde{g})$. We take

$$f = r^{\frac{2-n}{2}} \psi$$

for $\psi \in C_c^{0,1}(N, \tilde{g})$. Then

$$\tilde{\nabla} f = r^{\frac{2-n}{2}} \tilde{\nabla}\psi - \frac{n-2}{2} r^{-\frac{n}{2}} \psi \tilde{\nabla} r,$$

so

$$\begin{aligned} |\tilde{\nabla} f|_{\tilde{g}}^2 &= r^{2-n} |\tilde{\nabla}\psi|_{\tilde{g}}^2 + \frac{(n-2)^2}{4} r^{-n} \psi^2 |\tilde{\nabla} r|_{\tilde{g}}^2 \\ &\quad - (n-2) r^{1-n} \psi \langle \tilde{\nabla}\psi, \tilde{\nabla} r \rangle_{\tilde{g}} \\ &:= a + b + c. \end{aligned}$$

We have

$$\int_N r^{n-2} a\, d\tilde{\mu} = \int_N |\tilde{\nabla}\psi|^2_{\tilde{g}}\, d\tilde{\mu}.$$

Since

$$r^{-2}|\tilde{\nabla} r|^2_{\tilde{g}} = |dr|^2,$$

we have

$$\int_N r^{n-2} b\, d\tilde{\mu} = \int_N \frac{(n-2)^2}{4}|dr|^2\psi^2\, d\tilde{\mu}.$$

Finally, we use integration by parts and the displayed formula for $\tilde{\Delta}\log r$ below to compute

$$\begin{aligned}
\int_N r^{n-2} c\, d\tilde{\mu} &= -\int_N \frac{n-2}{2}\left\langle \tilde{\nabla}(\psi^2), \tilde{\nabla}(\log r)\right\rangle_{\tilde{g}} d\tilde{\mu} \\
&= \int_N \frac{n-2}{2}\tilde{\Delta}(\log r)\psi^2\, d\tilde{\mu} \\
&= \int_N \left(\frac{n(n-2)}{2} - \frac{n(n-2)}{2}|dr|^2\right)\psi^2\, d\tilde{\mu}.
\end{aligned}$$

Recall that we have the following

$$\tilde{\Delta}\log r = r^2(\Delta \log r - (n-2)r^{-2}|\nabla r|^2) = n - nr^{-2}|\nabla r|^2$$

Altogether, we obtain

$$\int_N |\tilde{\nabla}\psi|^2_{\tilde{g}}\, d\tilde{\mu} \geq \int_N \left(r^2|A|^2 - \frac{n(n-2)}{2} + \left(\frac{n(n-2)}{2} - \frac{(n-2)^2}{4}\right)|dr|^2\right)\psi^2\, d\tilde{\mu},$$

as desired. ◆

Proposition 5.2.7

For $3 \leq n \leq 5$, *we have*

$$-\tilde{\Delta} + \frac{2}{n-2}\widetilde{\mathrm{BiRic}} \geq \frac{2(-n^3+6n^2-4n-8)}{8(n-2)}$$

in the spectral sense. ♠

Proof Combining Proposition 5.2.6 with Proposition 5.2.5, for every $\psi \in C_c^{0,1}(N, \tilde{g})$ we get

$$\int_N \left(|\tilde{\nabla}\psi|^2_{\tilde{g}} + \frac{2}{n-2}\widetilde{\mathrm{BiRic}}\psi^2\right) d\tilde{\mu} \geq \int_N \left(C_0 + C_1|dr|^2\right)\psi^2\, d\tilde{\mu},$$

where

$$C_0 = \frac{2(3n-3)}{n-2} - \frac{n(n-2)}{2}, \qquad C_1 = -\frac{2(2n-1)}{n-2} + \frac{n(n-2)}{2} - \frac{(n-2)^2}{4}.$$

Since $0 \leq |dr|^2 \leq 1$ and $C_1 \leq 0$ for $3 \leq n \leq 5$, the right-hand side is bounded from below by

$$C_0 + C_1 = \frac{2(-n^3+6n^2-4n-8)}{8(n-2)}.$$

This proves the claimed spectral lower bound. ◆

5.2.2 Construction of the μ-Bubble

Theorem 5.2.8

Suppose (N^n, g)*,* $3 \leq n \leq 4$*, is a compact manifold with boundary* $\partial N = \partial_+ N \cup \partial_- N$*. Assume that there are* $0 < a \leq 2$ *and a positive smooth function* w *such that*

$$-a\Delta^N w + \operatorname{BiRic}_N w \geq w.$$

Suppose that the distance between $\partial_- N$ *and* $\partial_+ N$ *is bounded below by* 5π*. Then one can find a smooth* μ*-bubble* Σ *separating* $\partial_- N$ *from* $\partial_+ N$ *such that, for every* $\psi \in C_c^\infty(\Sigma)$*,*

$$\frac{4}{4-a}\int_\Sigma |\nabla^\Sigma \psi|^2 \geq \int_\Sigma \left(\frac{1}{2} - \operatorname{Ric}_\Sigma\right)\psi^2.$$

Equivalently, in the spectral sense,

$$-\frac{4}{4-a}\Delta^\Sigma + \operatorname{Ric}_\Sigma \geq \frac{1}{2}.$$

The constant 1 is only a normalization and can be replaced by any positive constant after rescaling the metric.

5.2.3 Second variation of weighted area functional

Lemma 5.2.9

The minimizer to the following functional

$$\mathcal{A}(\Omega) = \int_{\partial\Omega} w d\mathcal{H}^{n-1} - \int_\Omega h w d\mathcal{H}^n,$$

satisfies the following equation

$$\begin{aligned}&\int_\Sigma w\left|\nabla^\Sigma \phi\right|^2 + \frac{1}{2}R_\Sigma \phi^2 w - \phi\Delta^\Sigma w - \frac{1}{2}w^{-1}\left\langle \nabla^N w, \nu\right\rangle^2 \phi^2 - \int_\Sigma \phi^2(-\Delta^N w + \frac{1}{2}R_N w)\\ &-\int_\Sigma \left[\phi^2 w\left\langle \nabla^N h, \nu\right\rangle + \frac{1}{2}h^2\phi^2 w\right] \geq 0.\end{aligned}$$

for any smooth function ϕ *on* Σ*.*

Given a domain Ω in N, we do a variation Ω_t under vector field V and denote $\Sigma_t = \partial\Omega_t \backslash \partial_- N$ and $\Sigma = \Sigma_0$. We choose V such that $\nabla_V V = 0$ and $V = \phi\nu$ along Σ. (Note that in general, we do not have $V \perp \Sigma_t$ for every t.)

Lemma 5.2.10

The following first and second variation formulas hold:

$$\mathcal{A}' = \int_\Sigma \phi\left[\left\langle \nabla^N w, \nu\right\rangle + wH - wh\right], \textit{ Recall that } H = \sum_{i=1}^n \left\langle \nabla_{e_i} e_i, \nu\right\rangle$$

$$\begin{aligned}\mathcal{A}'' = &\int_\Sigma \phi^2(\Delta^N w - \Delta^\Sigma w) + \int_\Sigma w\left(\left|\nabla^\Sigma \phi\right|^2 - \left(|A_\Sigma|^2 + \operatorname{Ric}(\nu,\nu)\right)\phi^2\right)\\ &-\int_\Sigma \left(\phi^2 h\left\langle \nabla^N w, \nu\right\rangle + \phi^2 w\left\langle \nabla^N h, \nu\right\rangle\right).\end{aligned}$$

The first variation is straightforward.

$$\begin{aligned}\mathcal{A}' &= \int_\Sigma \left\langle \nabla^N w, V\right\rangle - w\mathrm{div}^\Sigma(V) - hw\left\langle V,\nu\right\rangle d\mathcal{H}^3\\ &= \int_\Sigma \left\langle \nabla^N w, V\right\rangle + w\mathrm{div}^\Sigma(V^T) + wH\left\langle V,\nu\right\rangle - hw\left\langle V,\nu\right\rangle d\mathcal{H}^{n-1}\end{aligned}$$

Hence, the stationary gives us the formula

$$H = h - w^{-1}\left\langle \nabla^N w,\nu\right\rangle$$

For the derivative of $\int_\Sigma w\,\mathrm{div}^\Sigma(V^T)\,d\mathcal{H}^{n-1}$, we rewrite it using integration by parts as

$$-\int_\Sigma \left\langle \nabla^\Sigma w, V^T\right\rangle.$$

After differentiating,

$$-\int_\Sigma \left\langle \nabla^\Sigma w, \nabla_V(V^T)\right\rangle = \int_\Sigma \left\langle \nabla^\Sigma w, \phi\nabla_V\nu\right\rangle$$

where we have used $\nabla_V V = 0$. Note that we have

$$\left\langle \nabla_V\nu, e_i\right\rangle = -\left\langle \nu, \nabla_V e_i\right\rangle = -\left\langle \nu, \nabla_{e_i}(\phi\nu)\right\rangle = -e_i(\phi) \implies \nabla_V\nu = -\nabla^\Sigma\phi$$

Then, the derivative of $\int_\Sigma \left\langle \nabla^\Sigma w, \phi\nabla_V\nu\right\rangle$ is

$$\int_\Sigma \left\langle \nabla^\Sigma w, \phi\nabla_V\nu\right\rangle = \int_\Sigma -\left\langle \nabla^\Sigma w, \nabla^\Sigma\phi\right\rangle\phi$$

Now, we can compute the second variation. Note that we can use $V\left\langle V,\nu\right\rangle = 0$.

$$\begin{aligned}\mathcal{A}'' = &\int_\Sigma \phi^2\mathrm{Hess}^N w(\nu,\nu) - \phi\left\langle \nabla^\Sigma w, \nabla^\Sigma\phi\right\rangle - \phi^2\left\langle \nabla^N w,\nu\right\rangle H\\ &-\int_\Sigma w\phi\left(\Delta^\Sigma\phi + (|A_\Sigma|^2 + \mathrm{Ric}(\nu,\nu))\phi\right) - \int_\Sigma\left(\phi^2 h\left\langle \nabla^N w,\nu\right\rangle + \phi^2 w\left\langle \nabla^N h,\nu\right\rangle\right).\end{aligned}$$

For the Hessian, we have

$$\begin{aligned}\mathrm{Hess}^N w(\nu,\nu) &= \mathrm{div}^N(\nabla^N w) - \mathrm{div}^\Sigma(\nabla^N w) = \Delta^N w - \Delta^\Sigma w - \mathrm{div}^\Sigma(\left\langle \nabla^N w,\nu\right\rangle\nu)\\ &= \Delta^N w - \Delta^\Sigma w + \left\langle \nabla^N w,\nu\right\rangle H\end{aligned}$$

using integration by parts, we have

$$\begin{aligned}\mathcal{A}'' = &\int_\Sigma \phi^2(\Delta^N w - \Delta^\Sigma w) + w\left(|\nabla^\Sigma\phi|^2 - (|A|^2 + \mathrm{Ric}_N(\nu,\nu))\phi^2\right)\\ &-\int_\Sigma\left(\phi^2 h\left\langle \nabla^N w,\nu\right\rangle + \phi^2 w\left\langle \nabla^N h,\nu\right\rangle\right).\end{aligned}$$

This is the weighted-area second variation formula used below.

5.2.4 Properties of the minimizer

Proposition 5.2.11

Suppose Ω is the minimizer to $\mathcal{A}$. Assume that the weight is $W = w^a$, where $0 < a \le 2$, $w > 0$, and

$$-a\Delta^N w + \mathrm{BiRic}_N\, w \ge w.$$

We also assume that the function h satisfies the following condition:

$$1 + h^2 - 2|\nabla^N h| \ge 0.$$

Then, for any $\psi \in C_c^\infty(\Sigma)$, we have

$$\frac{4}{4-a}\int_\Sigma |\nabla^\Sigma\psi|^2 \ge \int_\Sigma \psi^2\left(\frac{1}{2} - Ric_\Sigma\right)$$

We can also write it as

$$-\frac{4}{4-a}\Delta^{\Sigma}+Ric_{\Sigma}\geq\frac{1}{2}.$$

♠

Proof Suppose Ω is the minimizer to $\mathcal{A}$. In the spectral bi-Ricci application the weight in the functional is

$$W=w^{a},$$

and we consider

$$\phi=W^{-\frac{1}{2}}\psi=w^{-\frac{a}{2}}\psi.$$

Then

$$\nabla^{\Sigma}\phi=W^{-\frac{1}{2}}\nabla^{\Sigma}\psi-\frac{1}{2}W^{-\frac{1}{2}}\psi\nabla^{\Sigma}\log W.$$

Hence

$$\int_{\Sigma}W|\nabla^{\Sigma}\phi|^{2}=\int_{\Sigma}|\nabla^{\Sigma}\psi|^{2}+\frac{1}{4}\psi^{2}|\nabla^{\Sigma}\log W|^{2}-\psi\left\langle\nabla^{\Sigma}\psi,\nabla^{\Sigma}\log W\right\rangle.$$

The first variation gives

$$H=h-\left\langle\nabla^{N}\log W,\nu\right\rangle=h-a\left\langle\nabla^{N}\log w,\nu\right\rangle.$$

Put

$$v=\left\langle\nabla^{N}\log w,\nu\right\rangle.$$

Then $H=h-av$, and hence

$$ahv=\frac{1}{2}h^{2}+\frac{1}{2}a^{2}v^{2}-\frac{1}{2}H^{2}.$$

We also need to compare the ambient and intrinsic Laplacians of the weight $W=w^{a}$. Along Σ,

$$\begin{aligned}\frac{\Delta^{\Sigma}W}{W}&=a\frac{\Delta^{\Sigma}w}{w}+a(a-1)|\nabla^{\Sigma}\log w|^{2},\\ \frac{\Delta^{N}W}{W}&=a\frac{\Delta^{N}w}{w}+a(a-1)\left(|\nabla^{\Sigma}\log w|^{2}+v^{2}\right).\end{aligned}$$

Thus

$$\frac{\Delta^{N}W}{W}-\frac{\Delta^{\Sigma}W}{W}=a\left(\frac{\Delta^{N}w}{w}-\frac{\Delta^{\Sigma}w}{w}\right)+a(a-1)v^{2}.$$

Since Σ is stable for the weighted functional, the second variation formula with weight W gives, after moving the potential terms to the right-hand side,

$$\begin{aligned}&\int_{\Sigma}W\left(|\nabla^{\Sigma}\phi|^{2}-aw^{-1}\Delta^{\Sigma}w\,\phi^{2}\right)\\ \geq&\int_{\Sigma}\psi^{2}\left(-a\frac{\Delta^{N}w}{w}+|A|^{2}+\mathrm{Ric}_{N}(\nu,\nu)-\frac{1}{2}H^{2}+\frac{1}{2}h^{2}+\left\langle\nabla^{N}h,\nu\right\rangle\right)\\ &+\frac{a(2-a)}{2}\int_{\Sigma}\psi^{2}v^{2}.\end{aligned}\tag{5.2.2}$$

The final term is obtained by combining the Laplacian comparison contribution $-a(a-1)v^{2}$ with the ahv term. Since $0<a\leq 2$, it is nonnegative and can be discarded.

It remains to estimate the left-hand side of (5.2.2). After substituting $\phi=w^{-a/2}\psi$, it becomes

$$\int_{\Sigma}|\nabla^{\Sigma}\psi|^{2}+a\psi\left\langle\nabla^{\Sigma}\psi,\nabla^{\Sigma}\log w\right\rangle-\left(a-\frac{a^{2}}{4}\right)\psi^{2}|\nabla^{\Sigma}\log w|^{2},$$

and Young's inequality with $\varepsilon = (4-a)^{-1}$ gives the estimate. Therefore,

$$\int_\Sigma W\left(|\nabla^\Sigma \phi|^2 - aw^{-1}\Delta^\Sigma w\,\phi^2\right) \leq \frac{4}{4-a}\int_\Sigma |\nabla^\Sigma \psi|^2.$$

Combining this estimate with (5.2.2) and the spectral inequality $-a\Delta^N w + \mathrm{BiRic}_N w \geq w$ gives

$$\begin{aligned}\frac{4}{4-a}\int_\Sigma |\nabla^\Sigma \psi|^2 \geq& \int_\Sigma \psi^2\left(1 - \mathrm{BiRic}_N + |A|^2 + \mathrm{Ric}_N(\nu,\nu) - \frac{1}{2}H^2\right)\\ &+ \int_\Sigma \psi^2\left(\frac{1}{2}h^2 + \langle \nabla^N h, \nu\rangle\right).\end{aligned}$$

Now, we need to use $|A|^2 + \mathrm{Ric}_N(\nu,\nu)$ to bound BiRic_N.

Lemma 5.2.12

We have the following inequality:

$$|A|^2 + \mathrm{Ric}_N(\nu,\nu) \geq \mathrm{BiRic}_N - \mathrm{Ric}_\Sigma + \frac{6-n}{4}H^2.$$

♠

Remark 5.2.13. This is the extension of the usual Schoen–Yau trick. Recall that we have (for $n=3$)

$$|A|^2 + \mathrm{Ric}_N(\nu,\nu) = \frac{1}{2}R_N - \frac{1}{2}R_\Sigma + \frac{1}{2}|A|^2 + \frac{1}{2}H^2 \geq \frac{1}{2}R_N - \frac{1}{2}R_\Sigma + \frac{3}{4}H^2.$$

This is exactly the same as the inequality in the lemma.

Proof Suppose $e_1 \in T_p\Sigma$ is a unit direction where Ric_Σ attains its minimum. Using the Gauss equation, we have

$$\begin{aligned}Ric_\Sigma(e_1,e_1) &= \sum_{i=2}^{n-1} R^\Sigma_{1i1i}\\ &= \sum_{i=2}^{n-1} R^N_{1i1i} + A_{11}\sum_{i=2}^{n-1} A_{ii} - \sum_{i=2}^{n-1} A^2_{1i}.\end{aligned}$$

Note that we have

$$\begin{aligned}A_{11}\sum_{i=2}^{n-1} A_{ii} =& -A^2_{11} + A_{11}H = -A^2_{11} - H\sum_{i=2}^{n-1} A_{ii} + H^2\\ \geq& -A^2_{11} - \frac{1}{n-2}\left(\sum_{i=2}^{n-1} A_{ii}\right)^2 + \left(1 - \frac{n-2}{4}\right)H^2\\ \geq& -\sum_{i=1}^{n-1} A^2_{ii} + \frac{6-n}{4}H^2\end{aligned}$$

By the choice of e_1, $\mathrm{Ric}_\Sigma = \mathrm{Ric}_\Sigma(e_1,e_1)$. Hence, we have

$$\begin{aligned}\mathrm{Ric}_\Sigma + \mathrm{Ric}_N(\nu,\nu) &\geq \sum_{i=2}^{n-1} R^N_{1i1i} + \mathrm{Ric}_N(\nu,\nu) - |A|^2 + \frac{6-n}{4}H^2\\ &\geq \mathrm{BiRic}_N - |A|^2 + \frac{6-n}{4}H^2.\end{aligned}$$

Equivalently,

$$|A|^2 + \mathrm{Ric}_N(\nu,\nu) \geq \mathrm{BiRic}_N - \mathrm{Ric}_\Sigma + \frac{6-n}{4}H^2,$$

which is the desired inequality. ♦

Now, we go back to our inequality. When $3 \leq n \leq 4$, we have $\frac{6-n}{4} \geq \frac{1}{2}$ and therefore

$$\frac{4}{4-a}\int_\Sigma |\nabla^\Sigma \psi|^2 \geq \int_\Sigma \psi^2(1 - Ric_\Sigma) + \int_\Sigma \psi^2\left(\frac{1}{2}h^2 + \langle \nabla^N h, \nu\rangle\right).$$

Together with the condition for h, $1 + h^2 - 2|\nabla^N h| \geq 0$, we have

$$1 + \frac{1}{2}h^2 + \langle \nabla^N h, \nu \rangle \geq \frac{1}{2} + \frac{1}{2}(1 + h^2 - 2|\nabla^N h|) \geq \frac{1}{2}.$$

Hence

$$\frac{4}{4-a}\int_\Sigma |\nabla^\Sigma \psi|^2 \geq \int_\Sigma \psi^2 \left(\frac{1}{2} - Ric_\Sigma\right).$$

This is the desired inequality. ◆

5.2.5 Completion of the μ-Bubble Construction

To prove Theorem 5.2.8, choose h by

$$h(x) = -\frac{1}{2}\tan(\frac{\tilde{d}(x, \partial_- N)}{4} - \frac{\pi}{2})$$

where $\tilde{d}(x, \partial_- N)$ is a smoothing of $d_N(x, \partial_- N)$ with $\operatorname{Lip} \tilde{d} \leq 2$. We denote $\varphi(x) = \frac{\tilde{d}(x,\partial_- N)}{4} - \frac{\pi}{2}$. Then

$$|\nabla^N h| \leq \frac{1}{2}(1 + \tan^2\varphi), \qquad 2|\nabla^N h| \leq 1 + h^2.$$

To show the existence of the minimizer, fix a domain Ω_0 containing $\partial_- N$ but not $\partial_+ N$ such that $\partial\Omega_0$ lies in $\tilde{d}(x, \partial_- N) < 4\pi$. We consider minimizing the following (relative) energy functional

$$\mathcal{A}(\Omega) = \int_{\partial\Omega} w^a d\mathcal{H}^{n-1} - \int_{\Omega\setminus\Omega_0} hw^a d\mathcal{H}^n + \int_{\Omega_0\setminus\Omega} hw^a d\mathcal{H}^n$$

The direct method gives a minimizer in the corresponding relative homology class. Since $3 \leq n \leq 4$, the free boundary part $\Sigma = \partial\Omega \setminus \partial_- N$ is smooth after the usual regularity theory for prescribed-mean-curvature hypersurfaces. The choice $d_N(\partial_+ N, \partial_- N) \geq 5\pi$ gives enough room for the smoothing $\tilde{d}$ and hence for the above choice of h. Applying the stability inequality from the preceding proposition proves Theorem 5.2.8.

5.2.6 Diameter estimate and volume estimate under spectral Ricci curvature bound

The following result of Antonelli–Xu [AX24] gives the radius/diameter and volume estimates under a spectral Ricci lower bound. In this subsection $\mathrm{Ric}_N(v, v)$ denotes the Ricci tensor on a unit vector; when Ric_N appears without arguments in a spectral inequality, it means the smallest eigenvalue of the Ricci tensor. The bi-Ricci notation is the one fixed above. In the low-dimensional range considered here, the minimizers appearing in the μ-bubble arguments are smooth. The parameter α below denotes the coefficient of the spectral Laplacian term.

Theorem 5.2.14 (Antonelli–Xu spectral Bonnet–Myers and Bishop–Gromov)

Let (N^n, g) be a compact smooth Riemannian manifold, $3 \leq n \leq 5$, and let

$$0 \leq \alpha \leq \frac{n-1}{n-2}, \qquad \lambda > 0.$$

Suppose that there is a positive smooth function u such that

$$-\alpha\Delta u + \mathrm{Ric}_N\, u \geq (n-1)\lambda u. \tag{5.2.3}$$

Let $(\widetilde{N}, \tilde{g})$ be the universal cover and let $\tilde{u}$ be the lift of u. Then the diameter upper bound is

$$\operatorname{diam}(\widetilde{N}, \tilde{g}) \leq \frac{\pi}{\sqrt{\lambda}}\left(\frac{\max_N u}{\min_N u}\right)^{\frac{n-3}{n-1}\alpha}. \tag{5.2.4}$$

Moreover the sharp volume upper bound is

$$|\widetilde{N}| \leq \lambda^{-\frac{n}{2}}|S^n|, \tag{5.2.5}$$

where S^n denotes the unit round sphere. In particular $\pi_1(N)$ is finite. If equality holds in the volume estimate, then $\tilde{u}$ is constant and $\widetilde{N}$ is the round sphere of radius $\lambda^{-1/2}$. ♡

Corollary 5.2.15 (Complete case in the subcritical range)

Let (N^n, g) be complete, not assumed compact, with $3 \le n \le 5$. If $n > 3$ and $0 \le \alpha < \frac{4}{n-1}$, or if $n = 3$ and $0 \le \alpha \le 2$, and if (5.2.3) holds for some positive smooth u, then N is compact. Moreover $\pi_1(N)$ is finite and, for the universal cover $\widetilde{N}$,

$$\operatorname{diam}(\widetilde{N}, \tilde{g}) \le \frac{\pi}{\sqrt{\lambda}} \left(\frac{\max_N u}{\min_N u} \right)^{\frac{n-3}{n-1}\alpha}, \qquad |\widetilde{N}| \le \lambda^{-\frac{n}{2}} |S^n|.$$

In particular $|N| \le \lambda^{-n/2}|S^n|$. In addition there is a constant $C = C(n, \alpha)$ such that

$$\operatorname{diam}(N) \le C\lambda^{-1/2}.$$

♠

Proof This is precisely the complete-case corollary in [AX24], based on Xu's subcritical spectral Bonnet–Myers theorem. The latter gives compactness and the uniform bound $\operatorname{diam}(N) \le C(n,\alpha)\lambda^{-1/2}$ in the stated range. Once compactness is known, Theorem 5.2.14 applies and gives (5.2.4) and (5.2.5). Since N is the quotient of $\widetilde{N}$ by a finite group of deck transformations, the same volume upper bound also holds for N. ♦

Remark 5.2.16. Even in the larger sharp range $0 \le \alpha \le (n-1)/(n-2)$, if N is complete and the positive function u in (5.2.3) satisfies $0 < \inf_N u \le \sup_N u < \infty$, then Corollary 5.2.19 below proves compactness directly. Thus noncompact examples in the super-subcritical range must have u degenerating at infinity.

Weighted geodesic proof of the diameter estimates. We also record the weighted-geodesic calculation, since it gives another way to see the Bonnet–Myers part.

For a positive function u on (M^m, g), we define the weighted geodesic distance from p to q by

$$L_u^\alpha(p,q) = \inf_\eta \int_\eta u^\alpha \, ds_g,$$

where the infimum is taken over piecewise smooth curves from p to q. As above, Ric_M in a spectral inequality denotes the smallest eigenvalue of the Ricci tensor, while $\operatorname{Ric}_M(T,T)$ denotes the tensor on the vector T.

Lemma 5.2.17 (Spectral weighted Laplacian comparison)

Let (M^m, g) be complete, $m \ge 3$, and suppose that $u > 0$ satisfies

$$-\alpha\Delta u + \operatorname{Ric}_M u \ge (m-1)\lambda u, \qquad \lambda > 0. \tag{5.2.6}$$

Let $\alpha : [0, l] \to M$ be an L_u^α-minimizing curve from p to q, parametrized by g-arclength, and assume first that q is not a weighted cut point along α. Then the following hold in the barrier sense at q.

If $0 \le \alpha < 4/(m-1)$, then for every C^1 function ψ on $[0, l]$ with

$$\psi(0) = 0, \qquad \psi(l) = u(q)^{\alpha/2},$$

we have

$$\Delta_q L_u^\alpha(p,q) \le \int_0^l \left(C_{m,\alpha}\psi_s^2 - (m-1)\lambda\psi^2 \right) ds, \tag{5.2.7}$$

where

$$C_{m,\alpha} = (m-1) + \frac{\alpha(m-3)^2}{4\left(1 - \frac{m-1}{4}\alpha\right)}.$$

In particular, choosing

$$\psi(s) = u(q)^{\alpha/2} \sin\left(\frac{\pi s}{2l}\right)$$

gives

$$\Delta_q L_u^\alpha(p,q) \le u(q)^\alpha \left(\frac{C_{m,\alpha}\pi^2}{8l} - \frac{(m-1)\lambda l}{2}\right). \tag{5.2.8}$$

If $0 \le \alpha \le (m-1)/(m-2)$, *then for every* C^1 *function* ψ *with*

$$\psi(0) = 0, \qquad \psi(l) = u(q)^{\alpha/(m-1)},$$

we have

$$\Delta_q L_u^\alpha(p,q) \le \int_0^l u^{\frac{m-3}{m-1}\alpha}\left((m-1)\psi_s^2 - (m-1)\lambda\psi^2\right) ds. \tag{5.2.9}$$

♠

Proof Let $T = \alpha'$ and put

$$\tilde g = u^{2\alpha} g.$$

Then L_u^α is exactly the distance function of the conformal metric $\tilde g$. Write

$$\rho = L_u^\alpha(p,\cdot).$$

Along α we have, by the definition of $\tilde g$ and of the distance function ρ,

$$d\tilde s = u^\alpha\, ds, \qquad \tilde T = u^{-\alpha} T, \qquad \nabla^g L_u^\alpha = u^\alpha T.$$

We first compute $\tilde\Delta\rho$ in the conformal metric. Choose a $\tilde g$-parallel orthonormal frame along α,

$$\tilde E_1, \dots, \tilde E_{m-1}, \qquad \tilde E_m = \tilde T,$$

and write $\tilde E_i = u^{-\alpha} e_i$. Thus $e_1, \dots, e_{m-1}, T$ is g-orthonormal and $e_i \perp T$. Let ϕ be a test function with $\phi(0) = 0$ and $\phi(l) = 1$, and set

$$\tilde V_i = \phi \tilde E_i.$$

These fields vanish at p and equal $\tilde E_i(q)$ at q. Hence the ordinary second variation formula for the $\tilde g$-distance gives, for $1 \le i \le m-1$,

$$\tilde\nabla^2\rho(\tilde E_i, \tilde E_i)(q) \le \int_0^{\tilde l} \left((\tilde T\phi)^2 - \phi^2 \tilde R(\tilde T, \tilde E_i, \tilde T, \tilde E_i)\right) d\tilde s.$$

The missing m-th direction is radial and contributes nothing, since $\tilde\nabla^2\rho(\tilde T, \tilde T) = 0$ away from the cut locus. Summing the preceding inequality over the transverse directions gives

$$\tilde\Delta\rho(q) \le \int_0^l u^{-\alpha}\left((m-1)\phi_s^2 - \phi^2 \widetilde{\mathrm{Ric}}(T,T)\right) ds. \tag{5.2.10}$$

Here we used $d\tilde s = u^\alpha ds$, $\tilde T\phi = u^{-\alpha}\phi_s$, and $\widetilde{\mathrm{Ric}}(\tilde T, \tilde T) = u^{-2\alpha}\widetilde{\mathrm{Ric}}(T,T)$.

Now return to the original metric. The conformal Laplacian formula for $\tilde g = e^{2\log u^\alpha} g$ gives, at q,

$$\Delta^g \rho = u(q)^{2\alpha}\tilde\Delta\rho - (m-2)u(q)^\alpha (\log u^\alpha)_s(l). \tag{5.2.11}$$

Combining (5.2.10) and (5.2.11), we get

$$\begin{aligned}\Delta^g\rho(q) \le u^{2\alpha}(q)\int_0^l u^{-\alpha}\left((m-1)\phi_s^2 - \phi^2\widetilde{\mathrm{Ric}}(T,T)\right) ds \\ - (m-2)u(q)^\alpha(\log u^\alpha)_s(l).\end{aligned} \tag{5.2.12}$$

Now rewrite the curvature term in (5.2.12) in the original metric. The conformal Ricci formula is

$$\begin{aligned}\widetilde{\mathrm{Ric}} = \mathrm{Ric} - (m-2)\left(\nabla^2 \log u^\alpha - d\log u^\alpha \otimes d\log u^\alpha\right)\\ - \left(\Delta \log u^\alpha + (m-2)|\nabla \log u^\alpha|^2\right) g.\end{aligned}$$

The conformal connection formula is

$$\tilde{\nabla}_X Y = \nabla_X Y + X(\log u^\alpha)Y + Y(\log u^\alpha)X - g(X,Y)\nabla \log u^\alpha.$$

Applying it to the $\tilde{g}$-geodesic equation $\tilde{\nabla}_{\tilde{T}}\tilde{T} = 0$ gives

$$0 = u^{2\alpha}\tilde{\nabla}_{\tilde{T}}\tilde{T} = \nabla_T T + (\log u^\alpha)_s T - \nabla \log u^\alpha,$$

and therefore

$$\nabla_T T = \nabla^\perp \log u^\alpha, \qquad \nabla^2 \log u^\alpha(T,T) = (\log u^\alpha)_{ss} - |\nabla^\perp \log u^\alpha|^2.$$

Using this in the Ricci formula gives the pointwise identity along α

$$\begin{aligned}\widetilde{\mathrm{Ric}}(T,T) &= \mathrm{Ric}_M(T,T) - (m-2)\left((\log u^\alpha)_{ss} - \nabla_T T \cdot \nabla \log u^\alpha - (\log u^\alpha)_s^2\right)\\ &\quad - \left(\Delta \log u^\alpha + (m-2)|\nabla \log u^\alpha|^2\right)\\ &= \mathrm{Ric}_M(T,T) - (m-2)\left((\log u^\alpha)_{ss} - |\nabla^\perp \log u^\alpha|^2 - (\log u^\alpha)_s^2\right)\\ &\quad - \left(\Delta \log u^\alpha + (m-2)|\nabla \log u^\alpha|^2\right)\\ &= \mathrm{Ric}_M(T,T) - \Delta \log u^\alpha - (m-2)(\log u^\alpha)_{ss}. \end{aligned} \tag{5.2.13}$$

The endpoint term in (5.2.12) is written in the same $u(q)^{2\alpha}\int u^{-\alpha}(\cdots)\,ds$ scale as

$$-(m-2)u(q)^\alpha(\log u^\alpha)_s(l) = -u(q)^{2\alpha}\int_0^l \left((m-2)\phi^2 u^{-\alpha}(\log u^\alpha)_s\right)_s\,ds,$$

because $\phi(0) = 0$ and $\phi(l) = 1$. Substituting (5.2.13) into (5.2.12) and expanding this total derivative gives the original-metric formula

$$\begin{aligned}\Delta^g \rho(q) \le u(q)^{2\alpha}\int_0^l u^{-\alpha}\Big[(m-1)\phi_s^2 - 2(m-2)\phi\phi_s(\log u^\alpha)_s\\ + (m-2)\phi^2(\log u^\alpha)_s^2 - \mathrm{Ric}_M(T,T)\phi^2 + \phi^2\Delta \log u^\alpha\Big]\,ds.\end{aligned}$$

Using (5.2.6) and $\mathrm{Ric}_M(T,T) \ge \mathrm{Ric}_M$, along the curve

$$-\mathrm{Ric}_M(T,T) + \alpha\frac{\Delta u}{u} \le -(m-1)\lambda.$$

Together with

$$\Delta \log u^\alpha = \alpha\frac{\Delta u}{u} - \alpha\frac{|\nabla u|^2}{u^2},$$

this gives

$$\begin{aligned}\Delta^g \rho(q) \le u(q)^{2\alpha}\int_0^l u^{-\alpha}\Big[(m-1)\phi_s^2 - 2(m-2)\phi\phi_s(\log u^\alpha)_s\\ + (m-2)\phi^2(\log u^\alpha)_s^2 - (m-1)\lambda\phi^2 - \alpha\phi^2\frac{|\nabla u|^2}{u^2}\Big]\,ds.\end{aligned} \tag{5.2.14}$$

Only at this point do we change the test function. The subcritical estimate uses the substitution

$$\phi = \frac{u^{\alpha/2}}{u(q)^\alpha}\psi, \qquad \psi(0) = 0, \qquad \psi(l) = u(q)^{\alpha/2}.$$

Thus

$$\phi_s = \frac{u^{\alpha/2}}{u(q)^\alpha}\left(\psi_s + \frac{\alpha}{2}\psi\frac{u_s}{u}\right), \qquad u_s = T(u).$$

Inserting this into (5.2.14) cancels the weight u^α and gives

$$\Delta_q L_u^\alpha(p,q) \le \int_0^l \Big[(m-1)\psi_s^2 + \alpha(3-m)\psi\psi_s\frac{u_s}{u} + \frac{m-1}{4}\alpha^2\psi^2\frac{u_s^2}{u^2} - \alpha\psi^2\frac{|\nabla u|^2}{u^2} - (m-1)\lambda\psi^2\Big]\,ds.$$

Since $|\nabla u|^2 \ge u_s^2$, this implies

$$\Delta_q L_u^\alpha(p,q) \le \int_0^l \Big[(m-1)\psi_s^2 + \alpha(3-m)\psi\psi_s\frac{u_s}{u} - \alpha\left(1-\frac{m-1}{4}\alpha\right)\psi^2\frac{u_s^2}{u^2} - (m-1)\lambda\psi^2\Big]\,ds.$$

When $\alpha < 4/(m-1)$, Cauchy's inequality gives

$$\alpha(3-m)\psi\psi_s\frac{u_s}{u} - \alpha\left(1-\frac{m-1}{4}\alpha\right)\psi^2\frac{u_s^2}{u^2} \le \frac{\alpha(m-3)^2}{4\left(1-\frac{m-1}{4}\alpha\right)}\psi_s^2.$$

This proves (5.2.7); the sine choice gives (5.2.8).

For the bounded-u diameter estimate one uses a different exponent. Put

$$\phi = \frac{u^{\frac{m-2}{m-1}\alpha}}{u(q)^\alpha}\psi, \qquad \psi(0)=0, \qquad \psi(l) = u(q)^{\alpha/(m-1)}.$$

Then

$$\phi_s = \frac{u^{\frac{m-2}{m-1}\alpha}}{u(q)^\alpha}\left(\psi_s + \frac{m-2}{m-1}\alpha\psi\frac{u_s}{u}\right).$$

Substituting this into (5.2.14), the cross term cancels exactly and we obtain

$$\Delta_q L_u^\alpha(p,q) \le \int_0^l u^{\frac{m-3}{m-1}\alpha}\Big[(m-1)\psi_s^2 + \alpha\left(\frac{m-2}{m-1}\alpha - 1\right)\psi^2\frac{u_s^2}{u^2} - \alpha\psi^2\frac{|\nabla u|^2 - u_s^2}{u^2} - (m-1)\lambda\psi^2\Big]\,ds.$$

If $\alpha \le (m-1)/(m-2)$, the two terms involving derivatives of u are nonpositive. Dropping them gives (5.2.9). ♦

Corollary 5.2.18 (Subcritical finite diameter from weighted geodesics) *Let (M^m, g) be complete, $m \ge 3$, and suppose $u > 0$ satisfies (5.2.6). If*

$$0 \le \alpha < \frac{4}{m-1},$$

then

$$\operatorname{diam}(M) \le \pi\sqrt{\frac{C_{m,\alpha}}{(m-1)\lambda}}. \tag{5.2.15}$$

In particular M is compact. ♠

Proof Assume first that two points p, q are joined by an L_u^α-minimizing curve $\alpha : [0,l] \to M$ parametrized by g-arclength. Let $x = \alpha(l/2)$. The weighted excess

$$e(y) = L_u^\alpha(p,y) + L_u^\alpha(y,q) - L_u^\alpha(p,q)$$

has a local minimum at x. Using the two subsegments of α as barriers and applying (5.2.8) to each half, we get, in the viscosity sense,

$$0 \le \Delta e(x) \le u(x)^\alpha\left(\frac{C_{m,\alpha}\pi^2}{2l} - \frac{(m-1)\lambda l}{2}\right).$$

Therefore $l \le \pi\sqrt{C_{m,\alpha}/((m-1)\lambda)}$. Since $d_g(p,q) \le l$, this gives (5.2.15).

If the weighted minimizer is not realized, take a minimizing sequence. The usual Calabi limiting argument gives a broken minimizing object made of finite minimizing segments, weighted minimizing rays, and weighted minimizing lines. At the point where the corresponding excess vanishes, replace any ray by the point at parameter t on that ray and use the finite-segment barrier above. The right hand side contains the term $-(m-1)\lambda t/2$ for that ray and is negative for t sufficiently large, contradicting the local minimum of the excess. Thus the non-realized case cannot occur, and the same diameter bound holds for all pairs of points. Hopf–Rinow then implies compactness. ◆

Corollary 5.2.19 (Diameter bound with two-sided bounds for u)

Let (M^m, g) be complete, $m \geq 3$, and suppose $u > 0$ satisfies (5.2.6) with

$$0 \leq \alpha \leq \frac{m-1}{m-2}.$$

If

$$0 < u_- := \inf_M u \leq \sup_M u =: u_+ < \infty,$$

then

$$\operatorname{diam}(M) \leq \frac{\pi}{\sqrt{\lambda}} \left(\frac{u_+}{u_-}\right)^{\frac{m-3}{2(m-1)}\alpha}. \tag{5.2.16}$$

In particular M is compact. ♠

Proof Because u is bounded above and below, the weighted metric $u^{2\alpha}g$ is complete, so L_u^α-minimizers exist. Let $\alpha : [0, l] \to M$ be an L_u^α-minimizer from p to q, parametrized by g-arclength, and let $x = \alpha(l/2)$. As above, the weighted excess has a local minimum at x.

Use (5.2.9) on each half of α with

$$\psi(s) = u(x)^{\alpha/(m-1)} \sin\left(\frac{\pi s}{2(l/2)}\right).$$

Since $\alpha(m-3)/(m-1) \geq 0$, we estimate the positive term by $u_+^{\frac{m-3}{m-1}\alpha}$ and the negative term by $u_-^{\frac{m-3}{m-1}\alpha}$. Thus

$$0 \leq \Delta e(x) \leq (m-1)u(x)^{\frac{2\alpha}{m-1}} \left(\frac{u_+^{\frac{m-3}{m-1}\alpha}\pi^2}{2l} - \frac{\lambda u_-^{\frac{m-3}{m-1}\alpha} l}{2}\right).$$

Hence

$$l \leq \frac{\pi}{\sqrt{\lambda}} \left(\frac{u_+}{u_-}\right)^{\frac{m-3}{2(m-1)}\alpha}.$$

Since $d_g(p, q) \leq l$ and p, q were arbitrary, this proves (5.2.16). Compactness follows from Hopf–Rinow. ◆

Volume estimate.

Lemma 5.2.20 (Weighted profile differential inequality)

Assume the hypotheses of Theorem 5.2.14 and normalize $\min_{\tilde{N}} \tilde{u} = 1$. Set $\theta = 2\alpha/(n-1)$ and define, on $\tilde{N}$,

$$I(v) = \inf\left\{\int_{\partial E} \tilde{u}^\alpha : E \Subset \tilde{N}, \quad \int_E \tilde{u}^\theta = v\right\}.$$

Then I satisfies

$$I''I \leq -\frac{(I')^2}{n-1} - (n-1)\lambda \tag{5.2.17}$$

in the viscosity sense. ♠

Proof By the diameter part, $\widetilde{N}$ is compact; hence the minimizer E exists for every $v_0 \in (0, \int_{\widetilde{N}} \tilde{u}^\theta)$. We suppress tildes. Put

$$V(E) = \int_E u^\theta, \qquad A(E) = \int_{\partial E} u^\alpha.$$

Let $\Sigma = \partial E$ and take a smooth normal variation with variational field $\varphi\nu$. We extend φ to a neighborhood of Σ and use the same symbol for the extension. With the convention used here,

$$\partial_t d\mu_t = H\varphi\, d\mu, \qquad \partial_t \nu = -\nabla^\Sigma \varphi, \qquad \partial_t H = -\Delta^\Sigma \varphi - \big(|A_\Sigma|^2 + \mathrm{Ric}_N(\nu,\nu)\big)\varphi.$$

Also

$$\partial_t (u^{-1}u_\nu) = u^{-1}\mathrm{Hess}^N u(\nu,\nu)\varphi - u^{-1}\langle \nabla^\Sigma u, \nabla^\Sigma \varphi\rangle - u^{-2}u_\nu^2\varphi.$$

Thus

$$V'(0) = \int_\Sigma u^\theta \varphi, \qquad V''(0) = \int_\Sigma (H + \theta u^{-1}u_\nu)u^\theta\varphi^2 + u^\theta\varphi\varphi_\nu,$$

$$A'(0) = \int_\Sigma u^\alpha \varphi (H + \alpha u^{-1}u_\nu),$$

and differentiating this last expression gives the following formula. In the following displays, the same color marks the terms with the same origin: magenta terms cancel by integration by parts or by the normal derivative of $\varphi = u^{-\alpha}$; green terms are the spectral-Ricci pair; blue terms come from $|A_\Sigma|^2$; red terms come from u_ν^2; orange terms come from Hu_ν; and purple terms come from the surviving part of the last line.

$$\begin{aligned} A''(0) = &\int_\Sigma \Big(-\Delta^\Sigma\varphi - \mathrm{Ric}_N(\nu,\nu)\varphi - |A_\Sigma|^2\varphi \quad \Big) u^\alpha\varphi \\ &+ \int_\Sigma \begin{pmatrix} -\alpha u^{-2}u_\nu^2\varphi + \alpha u^{-1}\Delta^N u\,\varphi - \alpha u^{-1}\Delta^\Sigma u\,\varphi \\ -\alpha u^{-1}Hu_\nu\varphi - \alpha u^{-1}\langle\nabla^\Sigma u, \nabla^\Sigma\varphi\rangle \end{pmatrix} u^\alpha\varphi \\ &+ \int_\Sigma \begin{pmatrix} \alpha u^{\theta-1}u_\nu\varphi^2 + u^\theta\varphi\varphi_\nu \\ +Hu^\theta\varphi^2 \end{pmatrix} u^{\alpha-\theta}(H + \alpha u^{-1}u_\nu). \end{aligned} \tag{5.2.18}$$

The last line is just the derivative of the factor $u^\alpha\varphi\, d\mu_t$ in $A'(t)$, rewritten with a factor u^θ:

$$\partial_t(u^\alpha\varphi\, d\mu_t) = \begin{pmatrix} \alpha u^{\alpha-1}u_\nu\varphi^2 + u^\alpha\varphi\varphi_\nu \\ +Hu^\alpha\varphi^2 \end{pmatrix} d\mu.$$

The volume constraint implies that

$$u^{\alpha-\theta}(H + \alpha u^{-1}u_\nu) = A_v'(v_0),$$

where A_v is the area of this variation written as a function of V. Choose $\varphi = u^{-\alpha}$ and set

$$Q = \int_\Sigma u^{\theta-\alpha}, \qquad X = u^{\theta-\alpha}A_v'(v_0), \qquad Y = u^{-1}u_\nu.$$

Then $H = X - \alpha Y$. For $\varphi = u^{-\alpha}$, the tangential $\Delta^\Sigma u$ term cancels after integration by parts with the tangential gradient term

$$-\alpha u^{-1}\langle\nabla^\Sigma u, \nabla^\Sigma\varphi\rangle.$$

Hence the first two lines of (5.2.18), together with (5.2.3) and $|A_\Sigma|^2 \geq H^2/(n-1)$, are bounded above by

$$\int_\Sigma u^{-\alpha}\left[-\frac{H^2}{n-1} - \alpha HY - \alpha Y^2 - (n-1)\lambda\right].$$

The last line of (5.2.18) is

$$\int_\Sigma u^{-\alpha}XH,$$

because $\varphi_\nu = -\alpha u^{-\alpha} Y$ and $H + \alpha Y = X$. Therefore

$$A''(0) \le \int_\Sigma u^{-\alpha} \left[-\frac{H^2}{n-1} - \alpha HY - \alpha Y^2 - (n-1)\lambda + XH \right].$$

Expanding $H = X - \alpha Y$ gives

$$A''(0) \le \int_\Sigma u^{-\alpha} \Big[-\frac{X^2}{n-1} + \frac{2\alpha}{n-1} XY - \frac{\alpha^2}{n-1} Y^2 - \alpha Y^2 - \alpha XY + \alpha^2 Y^2 + X^2 - \alpha XY - (n-1)\lambda \Big].$$

The inverse-function chain rule gives

$$A_v''(v_0) = (V'(0))^{-2} A''(0) - (V'(0))^{-3} A'(0) V''(0).$$

Substituting the formulas for V', V'' and using $\theta = 2\alpha/(n-1)$ yields

$$\begin{aligned} Q^2 A_v''(v_0) &\le \int_\Sigma u^{-\alpha} \left[-\frac{X^2}{n-1} + \left(\frac{2\alpha}{n-1} - \theta \right) XY + \left(\frac{n-2}{n-1} \alpha^2 - \alpha \right) Y^2 - (n-1)\lambda \right] \\ &\le -\left(\frac{A_v'(v_0)^2}{n-1} + (n-1)\lambda \right) \int_\Sigma u^{2\theta - 3\alpha}. \end{aligned}$$

Here we used $\alpha \le (n-1)/(n-2)$ and $u \ge 1$. Holder's inequality gives

$$A_v(v_0) \int_\Sigma u^{2\theta - 3\alpha} \ge \left(\int_\Sigma u^{\theta - \alpha} \right)^2 = Q^2.$$

Thus

$$A_v(v_0) A_v''(v_0) \le -\frac{A_v'(v_0)^2}{n-1} - (n-1)\lambda.$$

Since A_v is an upper barrier for I at v_0, this is exactly the viscosity inequality (5.2.17). ♦

Lemma 5.2.21 (ODE comparison for the volume upper bound)

Let $V \in (0, \infty]$ *and let* $I : [0, V) \to \mathbb{R}$ *be continuous with* $I(0) = 0$ *and* $I(v) > 0$ *for* $v \in (0, V)$. *Suppose*

$$I'' I \le -\frac{(I')^2}{n-1} - (n-1)\lambda$$

in the viscosity sense on $(0, V)$, *and suppose*

$$\limsup_{v \to 0^+} v^{-\frac{n-1}{n}} I(v) \le n |B^n|^{1/n}.$$

Then

$$V \le \lambda^{-n/2} |S^n|.$$

♠

Proof Set

$$\psi = I^{\frac{n}{n-1}}.$$

Applying the chain rule to positive upper test functions for I, the preceding viscosity inequality is equivalent to

$$\psi'' \le -n\lambda \psi^{\frac{2-n}{n}}$$

in the viscosity sense. The small-volume assumption gives

$$\psi_+'(0) := \limsup_{v \to 0^+} \frac{\psi(v)}{v} \le n^{\frac{n}{n-1}} |B^n|^{\frac{1}{n-1}} = n |S^{n-1}|^{\frac{1}{n-1}}.$$

For $\zeta > 0$ define

$$\mu(r) = \frac{\sin(\sqrt{\lambda} r)}{\sqrt{\lambda}}, \qquad v_\zeta(r) = \zeta \int_0^r \mu(s)^{n-1}\, ds, \qquad 0 \le r \le \frac{\pi}{\sqrt{\lambda}},$$

and define the model profile I_ζ by

$$I_\zeta(v_\zeta(r)) = \zeta\mu(r)^{n-1}.$$

Let $\psi_\zeta = I_\zeta^{n/(n-1)}$. Since

$$\psi_\zeta(v_\zeta(r)) = \zeta^{\frac{n}{n-1}}\mu(r)^n, \qquad \frac{dv_\zeta}{dr} = \zeta\mu(r)^{n-1},$$

direct differentiation gives

$$\psi_\zeta'' = -n\lambda\psi_\zeta^{\frac{2-n}{n}}, \qquad (\psi_\zeta)'_+(0) = n\zeta^{\frac{1}{n-1}}.$$

The existence interval of I_ζ has length

$$V_\zeta = \zeta\int_0^{\pi/\sqrt{\lambda}} \mu(s)^{n-1}\,ds.$$

For $\zeta = |S^{n-1}|$, this is exactly $\lambda^{-n/2}|S^n|$.

Assume by contradiction that $V > \lambda^{-n/2}|S^n|$. Choose $\zeta > |S^{n-1}|$ so close to $|S^{n-1}|$ that $V_\zeta < V$. Then $(\psi_\zeta)'_+(0) > \psi'_+(0)$, so $\psi < \psi_\zeta$ on $(0, \delta)$ for some $\delta > 0$. If the two functions first meet at $a \in (0, V_\zeta)$, then on $(0, a)$ we have $\psi < \psi_\zeta$. Since the function $s \mapsto -n\lambda s^{(2-n)/n}$ is increasing on $(0, \infty)$, the difference $w = \psi - \psi_\zeta$ satisfies $w'' < 0$ in the viscosity sense on $(0, a)$. Thus w is strictly concave there. But $w(0) = w(a) = 0$ and $w < 0$ on $(0, a)$, which is impossible for a concave function. Hence $\psi < \psi_\zeta$ on $(0, V_\zeta)$. Since $\psi_\zeta(V_\zeta) = 0$ while $V_\zeta < V$ and $I > 0$ on $(0, V)$, continuity gives a contradiction at V_ζ. Therefore $V \le \lambda^{-n/2}|S^n|$. ♦

Proof of the volume part of Theorem 5.2.14 Normalize $\min_{\widetilde{N}} \tilde{u} = 1$ and set

$$V_0 = \int_{\widetilde{N}} \tilde{u}^\theta.$$

The profile I satisfies (5.2.17). Since $\tilde{u}$ attains its minimum at some point $\tilde{p}$, small geodesic balls centered at $\tilde{p}$ satisfy

$$\int_{B_r(\tilde{p})} \tilde{u}^\theta = |B^n|r^n + O(r^{n+1}), \qquad \int_{\partial B_r(\tilde{p})} \tilde{u}^\alpha = n|B^n|r^{n-1} + O(r^n).$$

Since these balls are admissible competitors for I, it follows that

$$\limsup_{v\to 0} v^{-\frac{n-1}{n}} I(v) \le n|B^n|^{1/n}.$$

Lemma 5.2.21 applied with $V = V_0$ therefore gives

$$V_0 \le \lambda^{-n/2}|S^n|.$$

Since $\tilde{u} \ge 1$, we have

$$|\widetilde{N}| \le \int_{\widetilde{N}} \tilde{u}^\theta = V_0 \le \lambda^{-n/2}|S^n|.$$

If equality holds, then $\int_{\widetilde{N}} \tilde{u}^\theta = |\widetilde{N}|$ and $\tilde{u} \ge 1$, hence $\tilde{u} \equiv 1$. The spectral inequality becomes the pointwise Ricci bound $\mathrm{Ric}_{\widetilde{N}} \ge (n-1)\lambda$, and the equality case in the classical Bishop–Gromov theorem gives that $\widetilde{N}$ is the round sphere of radius $\lambda^{-1/2}$. ♦

We need a comparison of distance.

Lemma 5.2.22

Let $\varphi : M \to \mathbb{R}^{n+1}$ be the immersion and $\tilde{g} = r^{-2}g$ and $g = \varphi^*(\delta)$ with $0 \in M$. Given two points $p, q \in M$ with $d_{\tilde{g}}(p, q) \le D$, we have $r(p) \le e^D r(q)$. ♠

Proof Let α be a curve joining p and q with length $D+\varepsilon$. Then, we have

$$\begin{aligned}\log r(p)-\log r(q) &= \int \frac{\langle \nabla r, \alpha'(t)\rangle}{r(\alpha(t))}dt \le \int \frac{|\alpha'(t)|_g}{r}dt \\ &= \int |\alpha'(t)|_{\tilde{g}} d\tilde{t} = D+\varepsilon.\end{aligned}$$

◆

Proof of the Euclidean volume-growth estimate

Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ be as in the theorem, where $n=3$ or $n=4$. We fix any point $x_0 \in M$ and suppose $x_0 = 0$ after a translation. Let $\tilde{g} = r^{-2}g$, where $r(x) = |x|$.

The preceding spectral estimates and μ-bubble constructions give the following dimension-dependent constants. There exist $L_n, A_n, D_n < \infty$ such that, whenever N is a compact collar in $(M, \tilde{g})$ with $d_{\tilde{g}}(\partial_- N, \partial_+ N) \ge L_n$, one can find a smooth μ-bubble component Σ which separates $\partial_- N$ from $\partial_+ N$, lies in $\tilde{B}_{L_n}(\partial_- N)$, and satisfies

$$|\Sigma|_{\tilde{g}} \le A_n, \qquad \operatorname{diam}_{\tilde{g}}(\Sigma) \le D_n.$$

For $n=3$ and $n=4$ this follows from the μ-bubble reduction in Theorem 5.2.8 together with the spectral Ricci estimates in §5.2.6.

For any $\rho > 0$, choose R large such that

$$d_{\tilde{g}}(\partial B^M_\rho, \partial B^M_R) \ge L_n.$$

Set $N = B^M_R \setminus B^M_\rho$, with $\partial_- N = \partial B^M_\rho$ and $\partial_+ N = \partial B^M_R$, and apply the preceding paragraph. If $x \in \Sigma$, then there is $y \in \partial B^M_\rho$ with $d_{\tilde{g}}(x,y) \le L_n$. By the distance comparison lemma,

$$r(x) \le e^{L_n} r(y) \le e^{L_n}\rho.$$

Therefore $\Sigma \subset B_{C_n\rho}$ in the Euclidean metric. Since $g = r^2\tilde{g}$, the induced measures on the $(n-1)$-dimensional hypersurface Σ satisfy $d\mu_g = r^{n-1} d\mu_{\tilde{g}}$, and hence

$$|\Sigma|_g \le (C_n\rho)^{n-1}|\Sigma|_{\tilde{g}} \le C_n \rho^{n-1}.$$

Since M is simply connected and has one end [CSZ97], the side of Σ containing $B^M_\rho(x_0)$ is a compact region. Let Ω_ρ denote this region; then $B^M_\rho(x_0) \subset \Omega_\rho$ and $\partial\Omega_\rho = \Sigma$. The Michael–Simon isoperimetric inequality for minimal submanifolds [MS73] gives

$$|\Omega_\rho|_g \le C_n |\Sigma|_g^{\frac{n}{n-1}}.$$

Consequently,

$$|B^M_\rho(x_0)| \le |\Omega_\rho|_g \le C_n |\Sigma|_g^{\frac{n}{n-1}} \le C_n \rho^n.$$

◆

5.2.7 Mazet's proof in the ambient space $\mathbb{R}^6$

Here $\mathbb{R}^6$ means the hypersurface dimension is 5. Mazet [Maz24] proves that every complete, connected, two-sided stable minimal immersion

$$M^5 \hookrightarrow \mathbb{R}^6$$

is flat. The proof follows the Chodosh–Li–Minter–Stryker strategy [CLMS25], but with one extra parameter in the curvature quantity. This extra parameter is the weighted bi-Ricci curvature.

Let (N^m, g) be a Riemannian manifold, and let $\{e_1, \dots, e_m\}$ be an orthonormal basis. For $\alpha \in \mathbb{R}$, Mazet defines the α-weighted bi-Ricci curvature by

$$\mathrm{BRic}_\alpha(e_1,e_2) := \sum_{i=2}^{m} R(e_1,e_i,e_i,e_1) + \alpha \sum_{j=3}^{m} R(e_2,e_j,e_j,e_2).$$

Its pointwise minimum is

$$\Lambda_\alpha(p) := \min_{\substack{e,f\in T_pN\\ |e|=|f|=1,\ \langle e,f\rangle=0}} \mathrm{BRic}_\alpha(e,f).$$

When $\alpha = 1$, this is the usual bi-Ricci curvature:

$$\mathrm{BRic}_1(e_1,e_2) = \mathrm{Ric}(e_1,e_1) + \mathrm{Ric}(e_2,e_2) - R(e_1,e_2,e_2,e_1).$$

Thus α lets one change the relative weight of the curvature directions which are tangent to the eventual μ-bubble.

The first step is the same conformal change as before. Remove the points where the immersion hits the origin and set

$$\tilde g = r^{-2} g, \qquad N := M \setminus F^{-1}(0).$$

The Gulliver–Lawson observation is that $(N,\tilde g)$ is complete. Mazet's main spectral estimate says that, in dimension 5, one can choose

$$a = \frac{11}{10}, \qquad \alpha = \frac{40}{43}, \qquad \delta = \frac{3}{10},$$

so that the stability inequality implies the following weighted bi-Ricci spectral lower bound:

$$\int_N |\tilde\nabla\varphi|^2\, d\tilde\mu \ge \frac{1}{a}\int_N V\varphi^2\, d\tilde\mu, \qquad V \ge \delta - \widetilde\Lambda_\alpha, \qquad \varphi\in C^1_c(N).$$

Equivalently,

$$\int_N \left(a|\tilde\nabla\varphi|^2 + (\widetilde\Lambda_\alpha - \delta)\varphi^2 \right) d\tilde\mu \ge 0.$$

So $(N,\tilde g)$ does not have a pointwise lower bound $\widetilde\Lambda_\alpha \ge \delta$, but it has the corresponding spectral lower bound.

The second step is to build a weighted μ-bubble in a long annulus of $(N,\tilde g)$. The bubble is a hypersurface

$$\Sigma^4 = \partial\Omega$$

which separates the inner and outer boundary of the annulus. The second variation of the weighted μ-bubble turns the spectral BRic_α bound on N into a spectral Ricci bound on Σ. More precisely, if

$$\lambda_\Sigma(x) := \min_{|v|=1} \mathrm{Ric}_\Sigma(v,v),$$

then the induced metric on Σ satisfies an inequality of the form

$$\int_\Sigma \left(\frac{4}{(4-a)\alpha}|\nabla\phi|^2 + \lambda_\Sigma\phi^2 \right) d\mu_\Sigma \ge \frac{\delta}{2\alpha}\int_\Sigma \phi^2\, d\mu_\Sigma.$$

For Mazet's parameters,

$$\frac{4}{(4-a)\alpha} = \frac{43}{29} < \frac{3}{2} = \frac{k-1}{k-2}, \qquad k = \dim\Sigma = 4.$$

This is the numerical point that allows the Antonelli–Xu spectral Bishop–Gromov theorem [AX24] to be applied to the 4-dimensional bubble. It gives a uniform $\tilde g$-volume bound for Σ.

Finally one returns to the original metric g. If Σ is chosen around $B^M_\rho(p_0)$, the conformal collar bound gives

$$r \le C\rho \qquad \text{on } \Sigma,$$

so the $\tilde g$-volume bound for Σ becomes a g-area bound

$$|\Sigma|_g \le C\rho^4.$$

Since M has one end, one can choose the relevant component of Σ to enclose $B^M_\rho(p_0)$. The Michael–Simon–Brendle isoperimetric inequality [MS73, Bre21] on minimal hypersurfaces then gives

$$|B^M_\rho(p_0)| \le C|\Sigma|_g^{5/4} \le C\rho^5.$$

This is the Euclidean volume growth needed by the Schoen–Simon–Yau stable Bernstein theorem. Hence M^5 is flat. In short, the new feature in $\mathbb{R}^6$ is that the weighted curvature $\mathrm{BRic}_{40/43}$ creates just enough spectral Ricci positivity on the 4-dimensional μ-bubble for Antonelli–Xu's volume estimate to close.

5.3 Green-kernel proof of the stable Bernstein theorem in $\mathbb{R}^4$

The Green-kernel proof of the stable Bernstein theorem in $\mathbb{R}^4$ is due to Cabré–Catino–Mari–Mastrolia–Roncoroni [CCM+26]. The key point is that this proof does not use μ-bubbles. Instead, it combines:

1. the positive supersolution furnished by stability;
2. the minimal positive Green kernel $\mathscr{G}$ of $-\Delta^M$;
3. a sharp pointwise estimate for $|\nabla\mathscr{G}|$ obtained from Bochner–Kato and a simple convexity lemma;
4. a weighted Schoen–Simon–Yau inequality;
5. a logarithmic cut-off in the variable $\mathscr{G}$.

Theorem 5.3.1 (Stable Bernstein in $\mathbb{R}^4$)

Let $M^3 \hookrightarrow \mathbb{R}^4$ be a connected, complete, two-sided, stable minimal immersion. Then M is an affine hyperplane. ♡

Reduction to bounded curvature. It is enough to prove the theorem under the auxiliary assumption

$$|A| \in L^\infty(M).$$

Indeed, if the theorem were false, then $|A|$ is nonzero somewhere. If $|A|$ is globally bounded, we are already in the bounded-curvature case. If not, one performs the standard point-picking argument on intrinsic balls $B^M_j(p_0)$: choose q_j and a scale $\lambda_j = |A(q_j)| \to \infty$ so that, after replacing the immersion by

$$F_j(x) = \lambda_j\big(F(x) - F(q_j)\big),$$

the rescaled hypersurfaces have $|A_j|(q_j) = 1$ and uniformly bounded curvature on larger and larger intrinsic balls around q_j. Stability and minimality are scale invariant, so a smooth local compactness theorem gives a complete, two-sided, stable minimal limit in $\mathbb{R}^4$ with bounded second fundamental form and $|A|(0) = 1$. The bounded-curvature case below would force this limit to be flat, a contradiction.

Green kernel and stability supersolution. Let now $M^n \hookrightarrow \mathbb{R}^{n+1}$ be complete, two-sided, stable, and minimal, with $n \ge 3$, and assume $|A| \in L^\infty(M)$. We keep n arbitrary until the last step. Stability means

$$\int_M |A|^2\phi^2\,d\mu \le \int_M |\nabla\phi|^2\,d\mu, \qquad \phi \in \mathrm{Lip}_c(M). \tag{5.3.1}$$

Equivalently, the operator $-\Delta - |A|^2$ is nonnegative. By the positive solution characterization of nonnegative Schrödinger operators, there exists a positive C^2 function u such that

$$\Delta u + |A|^2 u \le 0 \qquad \text{on } M. \tag{5.3.2}$$

We next explain why the Green kernel exists. The needed potential-theoretic input is the following.

Definition 5.3.2 (Parabolic and non-parabolic manifolds)

Let (N^m, g) *be a connected, complete, noncompact Riemannian manifold. We say that* N *is* ***non-parabolic*** *if the Laplacian* $-\Delta^N$ *admits a positive minimal Green kernel: for some, equivalently for every, point* $o \in N$ *there is a function* $G(o, \cdot) > 0$ *on* $N \setminus \{o\}$ *such that*

$$\Delta^N G(o, \cdot) = -\delta_o,$$

in the distributional sense, and G *is minimal among all positive fundamental solutions. If no such positive Green kernel exists,* N *is called* ***parabolic****.*

Equivalently, N *is parabolic if every positive superharmonic function on* N *is constant; equivalently, the Brownian motion on* N *is recurrent. Another useful equivalent criterion is the capacity criterion below.* ♣

Definition 5.3.3 (Capacity)

For a compact set $K \Subset N$*, define its 2-capacity by*

$$\mathrm{Cap}_2(K) := \inf\left\{ \int_N |\nabla\phi|^2 \, d\mu : \phi \in C_c^\infty(N), \quad \phi \geq 1 \textit{ on a neighborhood of } K \right\}.$$

Then N *is non-parabolic if and only if* $\mathrm{Cap}_2(K) > 0$ *for some nonempty compact set* K*; it is parabolic if and only if* $\mathrm{Cap}_2(K) = 0$ *for every compact* K*.* ♣

Proposition 5.3.4 (Sobolev inequality implies non-parabolic)

Let (N^m, g) *be complete and noncompact, with* $m > 2$*. Suppose that there is a constant* $S > 0$ *such that*

$$\left(\int_N |f|^{\frac{2m}{m-2}} \, d\mu \right)^{\frac{m-2}{m}} \leq S \int_N |\nabla f|^2 \, d\mu \qquad \forall f \in C_c^\infty(N). \tag{5.3.3}$$

Then N *is non-parabolic.* ♠

Proof Choose a compact set $K \Subset N$ with $|K| > 0$. If $\phi \in C_c^\infty(N)$ and $\phi \geq 1$ near K, then

$$|K| \leq \int_K |\phi|^{\frac{2m}{m-2}} \, d\mu \leq \int_N |\phi|^{\frac{2m}{m-2}} \, d\mu.$$

Raising both sides to the power $(m-2)/m$ and using (5.3.3), we get

$$|K|^{\frac{m-2}{m}} \leq \left(\int_N |\phi|^{\frac{2m}{m-2}} \, d\mu \right)^{\frac{m-2}{m}} \leq S \int_N |\nabla\phi|^2 \, d\mu.$$

Taking the infimum over all admissible ϕ gives

$$\mathrm{Cap}_2(K) \geq S^{-1} |K|^{\frac{m-2}{m}} > 0.$$

By the capacity criterion, N is non-parabolic. ♦

Proposition 5.3.5 (Minimal hypersurfaces of dimension $n > 2$ are non-parabolic)

Let $M^n \hookrightarrow \mathbb{R}^{n+1}$ *be a complete minimal hypersurface with* $n > 2$*. Then* M *is non-parabolic.* ♠

Proof We use the sharp Michael–Simon Sobolev inequality of Brendle [Bre21]; in the present minimal hypersurface case it gives the following $W^{1,2}$ Sobolev consequence:

$$\left(\int_M |f|^{\frac{2n}{n-2}} \, d\mu \right)^{\frac{n-2}{n}} \leq C_n \int_M |\nabla f|^2 \, d\mu.$$

Since $n > 2$, Proposition 5.3.4 applies and M is non-parabolic. ♦

For the theorem, Proposition 5.3.5 applies to our M^n, since $n \geq 3$. Hence there exists a minimal positive

Green kernel $\mathscr{G}$ of $-\Delta$ with pole $o \in M$:

$$\Delta \mathscr{G} = -\delta_o, \qquad \mathscr{G} > 0, \qquad \mathscr{G}(x) \to 0 \quad \text{as } x \to \infty.$$

For a.e. $s > 0$, the level set

$$\Sigma_s := \{\mathscr{G} = s\},$$

is smooth, and the Green flux identity gives

$$\int_{\Sigma_s} |\nabla \mathscr{G}| \, d\sigma = 1. \tag{5.3.4}$$

Indeed, applying the divergence theorem to $\{\mathscr{G} > s\} \setminus B_\varepsilon(o)$ and letting $\varepsilon \to 0$ gives exactly the mass of the pole.

Ricci lower bound from the Gauss equation. Let $\{e_i\}_{i=1}^n$ diagonalize A at a point, with principal curvatures κ_i and $\sum_i \kappa_i = 0$. The traced Gauss equation gives

$$\operatorname{Ric}(e_i, e_i) = \sum_{j \neq i} A_{ii} A_{jj} = -\kappa_i^2.$$

Thus, for every unit vector v,

$$\operatorname{Ric}(v, v) = -|A(v, \cdot)|^2.$$

Since A is trace-free,

$$\kappa_i^2 = \left(\sum_{j \neq i} \kappa_j \right)^2 \leq (n-1) \sum_{j \neq i} \kappa_j^2 = (n-1)(|A|^2 - \kappa_i^2),$$

and hence $\kappa_i^2 \leq \frac{n-1}{n} |A|^2$. Therefore

$$\operatorname{Ric} \geq -\frac{n-1}{n} |A|^2 g. \tag{5.3.5}$$

The key convexity lemma. We shall use the following elementary computation several times.

Lemma 5.3.6 (Subsolution from a subsolution and a supersolution)

Let $w, v > 0$ be C^2 functions on a domain $\Omega \subset M$ satisfying

$$\Delta w + W w \geq 0, \qquad \Delta v + V v \leq 0.$$

For $\delta \geq 0$, set

$$\xi_\delta = w^{1+\delta} v^{-\delta}.$$

Then

$$\Delta \xi_\delta \geq \big(\delta V - (1+\delta) W\big) \xi_\delta + \delta(1+\delta) \left| \nabla \log \frac{w}{v} \right|^2 \xi_\delta. \tag{5.3.6}$$

♠

Proof Since $\xi_\delta = e^{\log \xi_\delta}$,

$$\Delta \xi_\delta = \xi_\delta \left(\Delta \log \xi_\delta + |\nabla \log \xi_\delta|^2 \right).$$

Now

$$\log \xi_\delta = (1+\delta) \log w - \delta \log v.$$

Using

$$\Delta \log w = \frac{\Delta w}{w} - |\nabla \log w|^2, \qquad \Delta \log v = \frac{\Delta v}{v} - |\nabla \log v|^2,$$

we get

$$\begin{aligned}\Delta \log \xi_\delta &= (1+\delta)\Delta \log w - \delta \Delta \log v \\ &\geq \delta V - (1+\delta)W - (1+\delta)|\nabla \log w|^2 + \delta |\nabla \log v|^2.\end{aligned}$$

Also,

$$\begin{aligned}|\nabla \log \xi_\delta|^2 &= (1+\delta)^2|\nabla \log w|^2 + \delta^2 |\nabla \log v|^2 \\ &\quad - 2\delta(1+\delta)\left\langle \nabla \log w, \nabla \log v\right\rangle.\end{aligned}$$

Adding the last two displays gives

$$\Delta \log \xi_\delta + |\nabla \log \xi_\delta|^2 \geq \delta V - (1+\delta)W + \delta(1+\delta)\left|\nabla \log w - \nabla \log v\right|^2,$$

which is (5.3.6). ◆

Pointwise gradient estimate for the Green kernel. We first record the local differential inequality for the Green kernel.

Proposition 5.3.7 (Bochner–Kato subsolution for the Green kernel)

Let

$$q = |\nabla \mathscr{G}|, \qquad \alpha = \frac{n-2}{n-1}, \qquad w = q^\alpha.$$

On the open set $\{q>0\}\setminus\{o\}$, *the function* w *satisfies*

$$\Delta w + \frac{n-2}{n}|A|^2 w \geq 0. \tag{5.3.7}$$

♠

Proof Recall first the standard Bochner formula: for every smooth function f on a Riemannian manifold,

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \langle \nabla f, \nabla \Delta f\rangle + \mathrm{Ric}(\nabla f, \nabla f).$$

We apply this to $f = \mathscr{G}$ on $M\setminus\{o\}$. Since $\Delta\mathscr{G} = 0$ away from the pole, the middle term vanishes. Hence

$$\frac{1}{2}\Delta q^2 = |\nabla^2\mathscr{G}|^2 + \mathrm{Ric}(\nabla\mathscr{G}, \nabla\mathscr{G}).$$

The refined Kato inequality for harmonic functions gives

$$|\nabla^2\mathscr{G}|^2 \geq \frac{n}{n-1}|\nabla q|^2.$$

Together with the Ricci lower bound (5.3.5), this gives

$$\frac{1}{2}\Delta q^2 \geq \frac{n}{n-1}|\nabla q|^2 - \frac{n-1}{n}|A|^2 q^2.$$

Since

$$\frac{1}{2}\Delta q^2 = q\Delta q + |\nabla q|^2,$$

we subtract $|\nabla q|^2$ and obtain

$$q\Delta q \geq \frac{1}{n-1}|\nabla q|^2 - \frac{n-1}{n}|A|^2 q^2. \tag{5.3.8}$$

Now compute the Laplacian of $w = q^\alpha$:

$$\begin{aligned}\Delta w &= \alpha q^{\alpha-1}\Delta q + \alpha(\alpha-1)q^{\alpha-2}|\nabla q|^2 \\ &\geq \alpha q^{\alpha-2}\left[\frac{1}{n-1}|\nabla q|^2 - \frac{n-1}{n}|A|^2q^2\right] + \alpha(\alpha-1)q^{\alpha-2}|\nabla q|^2,\end{aligned}$$

where we used (5.3.8) in the second line. The coefficient of $|\nabla q|^2$ is

$$\alpha\left(\frac{1}{n-1} + \alpha - 1\right) = \alpha\left(\frac{1}{n-1} + \frac{n-2}{n-1} - 1\right) = 0.$$

Hence

$$\Delta w \geq -\alpha\frac{n-1}{n}|A|^2 q^\alpha = -\frac{n-2}{n}|A|^2 w,$$

which is exactly (5.3.7). ♦

Apply Lemma 5.3.6 to

$$W = \frac{n-2}{n}|A|^2, \qquad V = |A|^2, \qquad v = u, \qquad \delta \geq 0.$$

For

$$\xi_\delta = w^{1+\delta}u^{-\delta},$$

the coefficient of the zeroth-order $|A|^2\xi_\delta$ term in Lemma 5.3.6 is

$$\delta V - (1+\delta)W = \left[\delta - (1+\delta)\frac{n-2}{n}\right]|A|^2 = \frac{2\delta-(n-2)}{n}|A|^2.$$

Thus this coefficient is nonnegative when $\delta \geq \frac{n-2}{2}$, and it vanishes exactly at the critical choice

$$\delta = \frac{n-2}{2}.$$

With this choice, $1+\delta = \frac{n}{2}$, and

$$\xi := w^{n/2}u^{-(n-2)/2} = |\nabla\mathscr{G}|^{\frac{n(n-2)}{2(n-1)}}u^{-\frac{n-2}{2}}$$

is subharmonic on $\{q>0\}\setminus U$, where

$$U := \{\mathscr{G} > 1\}.$$

Because $|A|$ is bounded, (5.3.5) gives a lower Ricci bound, and the Cheng–Yau gradient estimate [CY75] applied outside U gives

$$|\nabla\mathscr{G}| \leq C\mathscr{G}.$$

We package the remaining two comparison-principle arguments into one lemma. First $\mathscr{G}$ is compared with the stability supersolution u; this auxiliary estimate is used only to remove the negative power of u and prove $\xi \to 0$ at infinity. Then ξ is compared directly with the harmonic barrier $\mathscr{G}$.

Lemma 5.3.8 (Comparison of ξ with the Green kernel) *There is a constant $C_0 > 0$ such that*

$$\xi \leq C_0\mathscr{G} \qquad \text{on } M\setminus U.$$

♠

Proof We first prove the auxiliary comparison

$$\mathscr{G} \leq Cu \qquad \text{on } M\setminus U.$$

Since $\mathscr{G} \to 0$ at infinity, the set $\bar{U} = \{\mathscr{G} \geq 1\}$ is compact. Choose C so large that

$$\mathscr{G} \leq Cu \qquad \text{on } \partial U.$$

This is possible because $\mathscr{G} = 1$ on ∂U and $u > 0$ there. Let Ω_R be a smooth exhaustion of M with $\bar{U} \Subset \Omega_R$, and let $\mathscr{G}_R$ be the Dirichlet Green function on Ω_R with pole o. On the annular region $\Omega_R\setminus\bar{U}$, the function $\mathscr{G}_R$ is harmonic and vanishes on $\partial\Omega_R$. Also,

$$\Delta(Cu) = C\Delta u \leq -C|A|^2u \leq 0,$$

so Cu is superharmonic. On the boundary of $\Omega_R\setminus\bar{U}$ we have

$$\mathscr{G}_R \leq \mathscr{G} = 1 \leq Cu \quad \text{on } \partial U, \qquad \mathscr{G}_R = 0 \leq Cu \quad \text{on } \partial\Omega_R.$$

The comparison principle on the bounded annulus therefore gives

$$\mathscr{G}_R \le Cu \qquad \text{on } \Omega_R \setminus U.$$

Letting $R \to \infty$ and using $\mathscr{G}_R \uparrow \mathscr{G}$, we obtain

$$\mathscr{G} \le Cu \qquad \text{on } M \setminus U.$$

This is where the first comparison is used. Combining it with $|\nabla \mathscr{G}| \le C\mathscr{G}$, we get

$$\xi \le C\mathscr{G}^{\frac{n(n-2)}{2(n-1)}} u^{-\frac{n-2}{2}} \le C\mathscr{G}^{\frac{n-2}{2(n-1)}} \to 0 \qquad \text{as } x \to \infty.$$

Now we perform the second comparison, this time between the subharmonic quantity ξ and a multiple of the harmonic Green kernel $\mathscr{G}$. Since $\bar{U}$ is compact and ξ is continuous up to ∂U, while $\mathscr{G} = 1$ on ∂U, we can choose C_0 so large that

$$\xi \le C_0 \mathscr{G} \qquad \text{on } \partial U.$$

Set

$$h := \xi - C_0 \mathscr{G}.$$

On $M \setminus U$, the pole o is not present, so $\mathscr{G}$ is harmonic. Also ξ is subharmonic there, in the weak sense obtained from the preceding pointwise computation by the standard regularization $q_\tau = (q^2 + \tau)^{1/2}$ and then letting $\tau \downarrow 0$. Hence

$$\Delta h = \Delta \xi - C_0 \Delta \mathscr{G} \ge 0 \qquad \text{on } M \setminus U.$$

Thus h is subharmonic. Moreover,

$$h \le 0 \qquad \text{on } \partial U.$$

We now explain how the boundary condition at infinity is used. Since $\xi(x) \to 0$ and $\mathscr{G}(x) \to 0$ as $x \to \infty$, for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \supset \bar{U}$ such that

$$h(x) = \xi(x) - C_0 \mathscr{G}(x) \le \xi(x) \le \varepsilon \qquad \text{on } M \setminus K_\varepsilon.$$

Choose a smooth bounded domain Ω_ε with $K_\varepsilon \Subset \Omega_\varepsilon$. On the bounded annular domain $\Omega_\varepsilon \setminus \bar{U}$, the maximum principle gives

$$h \le \max\{0, \varepsilon\} = \varepsilon.$$

Indeed, the inner boundary gives $h \le 0$, and the outer boundary lies in $M \setminus K_\varepsilon$, where $h \le \varepsilon$. Letting $\varepsilon \downarrow 0$, we obtain $h \le 0$ on $M \setminus U$, namely $\xi \le C_0 \mathscr{G}$. ♦

Taking the power $2/n$ yields the Green-kernel gradient estimate

$$|\nabla \mathscr{G}|^{\frac{n-2}{n-1}} \le C u^{\frac{n-2}{n}} \mathscr{G}^{\frac{2}{n}} \qquad \text{on } M \setminus U. \tag{5.3.9}$$

Weighted Schoen–Simon–Yau inequality. Set

$$\beta = \frac{n-2}{n}.$$

Simons' identity and the refined Kato inequality for A give, on $\{|A| > 0\}$,

$$|A|\Delta|A| \ge \frac{2}{n}|\nabla|A||^2 - |A|^4.$$

Let $\widetilde{w} = |A|^\beta$. Then

$$\begin{aligned} \Delta \widetilde{w} &= \beta|A|^{\beta-1}\Delta|A| + \beta(\beta-1)|A|^{\beta-2}|\nabla|A||^2 \\ &\ge \beta|A|^{\beta-2}\left[\frac{2}{n}|\nabla|A||^2 - |A|^4\right] + \beta(\beta-1)|A|^{\beta-2}|\nabla|A||^2. \end{aligned}$$

Again the gradient coefficient vanishes because

$$\frac{2}{n}+\beta-1=\frac{2}{n}+\frac{n-2}{n}-1=0.$$

Hence

$$\Delta\widetilde{w}+\frac{n-2}{n}|A|^2\widetilde{w}\ge 0.$$

For $0<t\le 1$, (5.3.2) gives

$$\Delta u^t = tu^{t-1}\Delta u + t(t-1)u^{t-2}|\nabla u|^2 \le -t|A|^2u^t.$$

Apply Lemma 5.3.6 to

$$w=\widetilde{w},\qquad v=u^t,\qquad W=\frac{n-2}{n}|A|^2,\qquad V=t|A|^2.$$

For $\delta\ge 0$, define

$$z=|A|^{\beta(1+\delta)}u^{-t\delta}.$$

Then

$$\Delta z\ge\left(t\delta-(1+\delta)\frac{n-2}{n}\right)|A|^2z.$$

Adding $|A|^2z$ to both sides,

$$\Delta z+|A|^2z\ge\gamma|A|^2z,\qquad \gamma:=\frac{2}{n}+\left(t-\frac{n-2}{n}\right)\delta. \tag{5.3.10}$$

Assume $\gamma>0$. Using (5.3.1) with test function $z\psi$ and integrating by parts gives

$$\begin{aligned}0&\le\int_M|\nabla(z\psi)|^2-|A|^2z^2\psi^2\\&=\int_M z^2|\nabla\psi|^2-\int_M z\psi^2(\Delta z+|A|^2z).\end{aligned}$$

Together with (5.3.10),

$$\gamma\int_M|A|^2z^2\psi^2\le\int_M z^2|\nabla\psi|^2.$$

Put

$$m:=1+\beta(1+\delta).$$

Taking $\psi=\varphi^m$ and using Young's inequality,

$$\int_M|A|^{2m}u^{-2t\delta}\varphi^{2m}\le C\int_M u^{-2t\delta}|\nabla\varphi|^{2m},\qquad \varphi\in\mathrm{Lip}_c(M). \tag{5.3.11}$$

Indeed, the right-hand side before Young is

$$C\int_M|A|^{2(m-1)}u^{-2t\delta}\varphi^{2m-2}|\nabla\varphi|^2,$$

and this is bounded by

$$\frac{1}{2}\int_M|A|^{2m}u^{-2t\delta}\varphi^{2m}+C\int_M u^{-2t\delta}|\nabla\varphi|^{2m}.$$

The parameter choice and the logarithmic cut-off. We now choose $\varphi=\eta(\mathscr{G})$. By the coarea formula,

$$\begin{aligned}\int_M u^{-2t\delta}|\nabla\varphi|^{2m}&=\int_M u^{-2t\delta}|\eta'(\mathscr{G})|^{2m}|\nabla\mathscr{G}|^{2m}\\&=\int_0^\infty|\eta'(s)|^{2m}\left[\int_{\Sigma_s}u^{-2t\delta}|\nabla\mathscr{G}|^{2m-1}\,d\sigma\right]ds.\end{aligned}\tag{5.3.12}$$

The estimate (5.3.9) gives

$$u^{-t\delta}|\nabla\mathscr{G}|^{\beta(1+\delta)}\le Cu^{-t\delta+\frac{(n-1)(n-2)}{n^2}(1+\delta)}\mathscr{G}^{\frac{2(n-1)}{n^2}(1+\delta)}.$$

We therefore impose

$$t\delta = \frac{(n-1)(n-2)}{n^2}(1+\delta).$$

Then

$$u^{-2t\delta}|\nabla\mathscr{G}|^{2m-1} = |\nabla\mathscr{G}|\left(u^{-t\delta}|\nabla\mathscr{G}|^{\beta(1+\delta)}\right)^2 \le C|\nabla\mathscr{G}|\mathscr{G}^{\frac{4(n-1)}{n^2}(1+\delta)}.$$

On Σ_s this becomes

$$u^{-2t\delta}|\nabla\mathscr{G}|^{2m-1} \le C|\nabla\mathscr{G}|s^{\frac{4(n-1)}{n^2}(1+\delta)}.$$

Using the flux identity (5.3.4), (5.3.12) becomes

$$\int_M u^{-2t\delta}|\nabla\eta(\mathscr{G})|^{2m} \le C\int_0^\infty |\eta'(s)|^{2m} s^{\frac{4(n-1)}{n^2}(1+\delta)}\,ds.$$

For a logarithmic cut-off to close, we need

$$\frac{4(n-1)}{n^2}(1+\delta) \ge 2m-1 = 1 + \frac{2(n-2)}{n}(1+\delta).$$

This inequality has a nonnegative solution δ only for $n=3$. In that case,

$$\beta = \frac{1}{3}, \qquad \delta = \frac{7}{2}, \qquad t = \frac{2}{7}, \qquad m = \frac{5}{2}, \qquad \gamma = \frac{1}{2}.$$

Then (5.3.11) reads

$$\int_M |A|^5 u^{-2}\varphi^5 \le C\int_M u^{-2}|\nabla\varphi|^5. \tag{5.3.13}$$

Also, the Green gradient estimate becomes

$$|\nabla\mathscr{G}|^{1/2} \le Cu^{1/3}\mathscr{G}^{2/3}.$$

Therefore

$$u^{-1}|\nabla\mathscr{G}|^{3/2} \le C\mathscr{G}^2,$$

and, on Σ_s,

$$u^{-2}|\nabla\mathscr{G}|^4 = |\nabla\mathscr{G}|\left(u^{-1}|\nabla\mathscr{G}|^{3/2}\right)^2 \le C|\nabla\mathscr{G}|s^4.$$

Hence

$$\int_M u^{-2}|\nabla\eta(\mathscr{G})|^5 \le C\int_0^\infty |\eta'(s)|^5 s^4\,ds.$$

For $R>1$, choose the logarithmic cut-off

$$\eta_R(s) = \begin{cases} 0, & 0 \le s \le R^{-2}, \\ 2 + \dfrac{\log s}{\log R}, & R^{-2} < s < R^{-1}, \\ 1, & s \ge R^{-1}. \end{cases}$$

Then $|\eta_R'(s)| = (s\log R)^{-1}$ on (R^{-2}, R^{-1}) and 0 elsewhere. Putting $\varphi = \eta_R(\mathscr{G})$ in (5.3.13),

$$\begin{aligned}\int_{\{\mathscr{G}\ge R^{-1}\}} |A|^5u^{-2} &\le C\int_M u^{-2}|\nabla\eta_R(\mathscr{G})|^5 \\ &\le C\int_{R^{-2}}^{R^{-1}} \frac{s^4}{s^5(\log R)^5}\,ds \\ &= \frac{C}{(\log R)^5}\int_{R^{-2}}^{R^{-1}}\frac{ds}{s} = \frac{C}{(\log R)^4}.\end{aligned}$$

Letting $R\to\infty$, the sets $\{\mathscr{G}\ge R^{-1}\}$ exhaust M up to the end, and the right-hand side tends to zero. Hence

$$|A|^5u^{-2} \equiv 0.$$

Since $u > 0$, we conclude $A \equiv 0$. Thus the immersion is totally geodesic, and the connected complete image is an affine hyperplane in $\mathbb{R}^4$. This proves Theorem 5.3.1.

Chapter 6 Applications to Positive Scalar Curvature and General Relativity

We now turn to applications where stable minimal hypersurfaces are used as probes of scalar curvature. The guiding mechanism is the Schoen–Yau dimension-reduction argument [SY79a, SY79b]: a stable minimal hypersurface inside a manifold with a scalar-curvature lower bound inherits, after a conformal change, a related lower bound in one lower dimension. This simple idea has two closely connected roles. In positive scalar curvature it leads to topological obstructions such as Geroch's conjecture. In mathematical general relativity it becomes a tool for proving mass inequalities, beginning with the positive mass theorem and, in the negatively curved setting, the Horowitz–Myers conjecture.

We first record the conformal calculation behind the descent argument, then explain how it proves the positive-scalar-curvature obstruction for tori. The last two sections reinterpret the same method in general relativity: the asymptotically flat case leads to the positive mass theorem, while the asymptotically locally hyperbolic toroidal case leads to the Horowitz–Myers mass inequality. A convenient modern reference for the positive-scalar-curvature part is Chodosh's notes Stable minimal surfaces and positive scalar curvature.

6.1 The conformal Laplacian

Let (X^n, g) be a closed Riemannian manifold, $n \geq 3$. We use the conformal Laplacian

$$L_g := -c_n \Delta^g + R_g, \qquad c_n := \frac{4(n-1)}{n-2}.$$

Here R_g is the scalar curvature of g. With our sign convention,

$$\int_X u L_g u \, d\mu_g = \int_X c_n |\nabla u|^2 + R_g u^2 \, d\mu_g.$$

If

$$\widehat{g} = u^{\frac{4}{n-2}} g, \qquad u > 0,$$

then this is the same as writing $\widehat{g} = e^{2f} g$ with

$$e^{2f} = u^{\frac{4}{n-2}}, \qquad f = \frac{2}{n-2} \log u.$$

For a general conformal change $\widehat{g} = e^{2f} g$, the scalar curvature formula is

$$R_{\widehat{g}} = e^{-2f} \big(R_g - 2(n-1)\Delta^g f - (n-1)(n-2)|\nabla f|^2 \big).$$

We now plug in $f = \frac{2}{n-2} \log u$. First,

$$\nabla f = \frac{2}{n-2} \frac{\nabla u}{u}, \qquad |\nabla f|^2 = \frac{4}{(n-2)^2} \frac{|\nabla u|^2}{u^2},$$

and

$$\Delta^g f = \frac{2}{n-2} \Delta^g \log u = \frac{2}{n-2} \left(\frac{\Delta^g u}{u} - \frac{|\nabla u|^2}{u^2} \right).$$

Therefore

$$\begin{aligned} R_{\widehat{g}} &= u^{-\frac{4}{n-2}} \left[R_g - \frac{4(n-1)}{n-2} \left(\frac{\Delta^g u}{u} - \frac{|\nabla u|^2}{u^2} \right) - \frac{4(n-1)}{n-2} \frac{|\nabla u|^2}{u^2} \right] \\ &= u^{-\frac{4}{n-2}} \left(R_g - \frac{4(n-1)}{n-2} \frac{\Delta^g u}{u} \right). \end{aligned}$$

The two $|\nabla u|^2/u^2$ terms cancel exactly: the first comes from $-2(n-1)\Delta^g f$, and the second comes from $-(n-1)(n-2)|\nabla f|^2$. Since $c_n = \frac{4(n-1)}{n-2}$, we get

$$R_{\widehat{g}} = u^{-\frac{n+2}{n-2}}\left(-c_n\Delta^g u + R_g u\right) = u^{-\frac{n+2}{n-2}} L_g u. \tag{6.1.1}$$

Lemma 6.1.1 (Positive first eigenvalue gives positive scalar curvature)

If the first eigenvalue of L_g is positive, then X admits a metric of positive scalar curvature in the conformal class of g.

Proof Let $u > 0$ be the first eigenfunction:

$$L_g u = \lambda_1 u, \qquad \lambda_1 > 0.$$

For $\widehat{g} = u^{4/(n-2)} g$, formula (6.1.1) gives

$$R_{\widehat{g}} = \lambda_1 u^{-\frac{4}{n-2}} > 0.$$

Lemma 6.1.2 (Nonnegative scalar curvature, positive somewhere)

Let X^n be closed, $n \geq 3$. If $R_g \geq 0$ and $R_g > 0$ somewhere, then X admits a metric of positive scalar curvature.

Proof We show that $\lambda_1(L_g) > 0$. For every nonzero ϕ,

$$\int_X \phi L_g \phi \, d\mu_g = \int_X c_n |\nabla\phi|^2 + R_g \phi^2 \, d\mu_g \geq 0.$$

Thus $\lambda_1 \geq 0$. If $\lambda_1 = 0$, let $u > 0$ be a first eigenfunction. Then

$$0 = \int_X c_n |\nabla u|^2 + R_g u^2 \, d\mu_g.$$

Both terms are nonnegative, so $\nabla u \equiv 0$ and $R_g u^2 \equiv 0$. Since $u > 0$, this forces $R_g \equiv 0$, contradicting the assumption that $R_g > 0$ somewhere. Hence $\lambda_1 > 0$, and Lemma 6.1.1 applies.

6.2 Conformal descent on a stable minimal hypersurface

Let (N^{n+1}, g_N) be a closed Riemannian manifold and let $\Sigma^n \subset N$ be a closed, two-sided, stable minimal hypersurface. Denote the induced metric on Σ by g. The stability inequality is

$$\int_\Sigma |\nabla\phi|^2 \, d\mu_g \geq \int_\Sigma \left(|A|^2 + \mathrm{Ric}_N(\nu,\nu)\right)\phi^2 \, d\mu_g, \qquad \phi \in C^\infty(\Sigma). \tag{6.2.1}$$

The Gauss equation, using $H = 0$, gives

$$R_N = R_\Sigma + 2\mathrm{Ric}_N(\nu,\nu) + |A|^2. \tag{6.2.2}$$

Combining (6.2.1) and (6.2.2),

$$\int_\Sigma 2|\nabla\phi|^2 + R_\Sigma \phi^2 \, d\mu_g \geq \int_\Sigma (R_N + |A|^2)\phi^2 \, d\mu_g. \tag{6.2.3}$$

This is the basic Schoen–Yau rearrangement.

Proposition 6.2.1 (Conformal descent)

Assume $R_N > 0$ and $n \geq 3$. Then every closed, two-sided, stable minimal hypersurface $\Sigma^n \subset N^{n+1}$ admits a metric of positive scalar curvature.

Proof Since Σ is compact and $R_N > 0$ on N, there is a number $\kappa > 0$ such that $R_N \geq \kappa$ on Σ. From (6.2.3),

$$\int_\Sigma 2|\nabla\phi|^2 + R_\Sigma\phi^2\, d\mu_g \geq \kappa \int_\Sigma \phi^2\, d\mu_g.$$

Because $c_n = 4(n-1)/(n-2) \geq 2$, we also have

$$\int_\Sigma c_n|\nabla\phi|^2 + R_\Sigma\phi^2\, d\mu_g \geq \kappa \int_\Sigma \phi^2\, d\mu_g.$$

Thus the first eigenvalue of the conformal Laplacian L_g is positive. The claim follows from Lemma 6.1.1. ♦

Remark 6.2.2 (The surface case). When $n = 2$, the same rearrangement gives useful topological information directly. Taking $\phi \equiv 1$ in (6.2.3) gives

$$\int_\Sigma R_\Sigma\, d\mu_g = 2\int_\Sigma K_\Sigma\, d\mu_g > 0.$$

Hence $\chi(\Sigma) > 0$ by Gauss–Bonnet. In particular a closed orientable stable minimal surface in a three-manifold with $R_N > 0$ is a union of two-spheres.

6.3 Geroch's Conjecture by Dimension Reduction

The following theorem is the form of the Geroch conjecture used in the positive mass theorem. The dimension restriction comes only from the regularity theory for area-minimizing hypersurfaces: an area-minimizing hypersurface in an n-manifold is smooth when $n \leq 7$.

Theorem 6.3.1 (Geroch conjecture, Schoen–Yau)

Let $3 \leq n \leq 7$. Let X^n be a closed oriented smooth manifold. If there is a continuous map

$$F : X \to \mathbb{T}^n$$

of nonzero degree, then X admits no metric of positive scalar curvature. ♡

Proof We argue by induction on n. The base $n = 2$ is Gauss–Bonnet: if a closed oriented surface maps to $\mathbb{T}^2$ with nonzero degree, then it has genus at least one, so it cannot carry a metric with positive Gaussian curvature.

Assume now $3 \leq n \leq 7$, and suppose the theorem is known in dimension $n-1$. Let g be a positive-scalar-curvature metric on X. Write

$$\theta_1, \ldots, \theta_n \in H^1(\mathbb{T}^n; \mathbb{Z})$$

for the standard degree-one cohomology classes, and set

$$\omega_i := F^*\theta_i \in H^1(X; \mathbb{Z}).$$

Since $\deg F \neq 0$,

$$\omega_1 \smile \cdots \smile \omega_n \neq 0 \qquad \text{in } H^n(X; \mathbb{Z}).$$

Let $\alpha = \omega_n$. Its Poincare dual is a nonzero integral homology class in $H_{n-1}(X; \mathbb{Z})$. Choose an area-minimizing integral current Σ in this class. Since $n \leq 7$, regularity theory implies that Σ is a smooth embedded closed hypersurface, possibly with several components and integer multiplicities. It is oriented, two-sided, minimal, and stable.

By Proposition 6.2.1, every component of Σ of dimension at least three admits a positive-scalar-curvature metric; in the surface case, Remark 6.2.2 says that every component is a two-sphere.

Now compute the cup product on Σ. Since $[\Sigma]$ is Poincare dual to ω_n,

$$\int_\Sigma \omega_1 \smile \cdots \smile \omega_{n-1} = \int_X \omega_1 \smile \cdots \smile \omega_n \neq 0.$$

Therefore at least one connected component Σ_0 has

$$\int_{\Sigma_0} \omega_1 \smile \cdots \smile \omega_{n-1} \neq 0.$$

Equivalently, the map

$$F_0 := (F_1, \dots, F_{n-1})|_{\Sigma_0} : \Sigma_0 \to \mathbb{T}^{n-1}$$

has nonzero degree.

If $n-1=2$, this contradicts the fact that Σ_0 is a two-sphere. If $n-1 \geq 3$, then Σ_0 carries a positive-scalar-curvature metric by conformal descent, contradicting the induction hypothesis in dimension $n-1$. This proves the theorem. ♦

Corollary 6.3.2

For $3 \leq n \leq 7$ and for every closed oriented n-manifold Y, the connected sum

$$\mathbb{T}^n \# Y$$

admits no metric of positive scalar curvature. ♠

Proof There is a degree-one map

$$\mathbb{T}^n \# Y \to \mathbb{T}^n$$

obtained by collapsing Y minus a ball to the connected-sum point and using the identity map on the torus side. Theorem 6.3.1 applies. ♦

6.4 The Positive Mass Theorem and the Reduction to Geroch's Conjecture

We now explain how the Riemannian positive mass theorem is reduced to Corollary 6.3.2. We state the time-symmetric version, which is the one governed by scalar curvature.

Remark 6.4.1 (Historical guide to the positive mass theorem). The time-symmetric, or Riemannian, positive mass theorem was first proved by Schoen–Yau by the minimal-hypersurface method [SY79b, SY81]. In the smooth dimension-reduction form of their argument, the regularity theory for area-minimizing hypersurfaces gives the theorem for $3 \leq n \leq 7$. Witten then gave a spinorial proof for spin asymptotically flat manifolds, without this regularity dimension restriction [Wit81].

The later history is largely about removing the non-spin dimension restriction. Schoen–Yau proposed an all-dimensional singular minimal-slicing approach [SY19], while Lohkamp's cut-off/compactification observation relates the Riemannian PMT to Geroch torus rigidity; this is the reduction used below. The generic-regularity work culminating in Chodosh–Mantoulidis–Schulze–Wang pushes the minimal-hypersurface/Lohkamp route to dimension 11 [CMSW25]. Bi–Hao–He–Shi–Zhu then proved the Riemannian PMT up to dimension 19 [BHH$^+$26], and Brendle–Wang subsequently gave a dimension descent scheme which closes the Riemannian theorem in arbitrary dimension [BW26a]. They also derived the spacetime positive energy theorem in arbitrary dimension from the Riemannian theorem and Jang-type arguments [BW26b].

There are also alternative proofs and stability directions which are useful to keep mentally separate from the reduction below. In dimension three, Bray–Kazaras–Khuri–Stern gave a harmonic-function proof of the

Riemannian PMT [BKKS22]. Stability versions ask whether small ADM mass forces the geometry to be close to Euclidean space. Lee–Sormani proved pointed intrinsic-flat stability for rotationally symmetric asymptotically flat manifolds [LS14]. Huang–Lee–Sormani proved pointed intrinsic-flat stability for graphical hypersurfaces in Euclidean space under natural technical hypotheses [HLS17]. Dong–Song proved a three-dimensional stability theorem in the general asymptotically flat setting: if the mass of a chosen end tends to zero, then after removing subsets whose boundary areas tend to zero, the remaining spaces converge to Euclidean 3-space in the pointed measured Gromov–Hausdorff topology [DS25]. The proof below records the classical smooth Schoen–Yau/Lohkamp/Geroch mechanism, because that is the version whose topology is most visible from stable minimal hypersurfaces.

Theorem 6.4.2 (Riemannian positive mass theorem, Schoen–Yau)

Let $3 \le n \le 7$. Let (M^n, g) be a complete one-ended asymptotically flat Riemannian manifold with nonnegative scalar curvature. Then the ADM mass is nonnegative. If the mass is zero and the usual rigidity hypotheses hold, then (M, g) is isometric to Euclidean space.

For a manifold with several asymptotically flat ends, the same conclusion is applied to one end at a time after the standard reduction which compactifies or fills the other ends without changing the sign of the chosen mass. We focus on the one-ended case because it contains the topological argument.

Recall the ADM mass of an end with asymptotically flat coordinates x:

$$m_{\mathrm{ADM}}(g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj})\nu^i \, d\sigma.$$

The nonnegativity part is the one whose topology is most transparent.

Definition 6.4.3 (AF coordinates and weighted Sobolev decay)

Fix an asymptotically flat end and write its coordinate chart as

$$x = (x^1, \dots, x^n) : M \setminus K \to \mathbb{R}^n \setminus B_R, \qquad r = |x|.$$

The symbol δ denotes the Euclidean metric in these coordinates. For a tensor T on the end and a number $\tau > 0$, we use

$$\|T\|^p_{L^p_{-\tau}} := \int_{M\setminus K} \big|(1+r)^{\tau} T\big|^p (1+r)^{-n}\, dx,$$

and

$$\|T\|_{W^{k,p}_{-\tau}} := \sum_{j=0}^{k} \|\partial^j T\|_{L^p_{-\tau-j}}.$$

Thus $T \in W^{k,p}_{-\tau}$ means that T decays like $r^{-\tau}$ and its j-th coordinate derivatives decay like $r^{-\tau-j}$ in weighted L^p sense.

In particular, an AF metric of Sobolev type (p, q) means

$$g_{ij} - \delta_{ij} \in W^{2,p}_{-q}, \qquad p > n, \qquad q > \frac{n-2}{2},$$

together with the integrability condition $R_g \in L^1$ when the ADM mass is used. The condition $p > n$ gives enough Sobolev embedding to read the metric and first derivatives pointwise, while $q > \frac{n-2}{2}$ is the decay threshold which makes the ADM mass stable under the density deformation.

Remark 6.4.4 (How this follows from the usual pointwise AF definition). If one starts instead from the classical pointwise AF assumption of order τ,

$$\partial^\alpha(g_{ij} - \delta_{ij}) = O(r^{-\tau-|\alpha|}), \qquad |\alpha| \le 2,$$

then the weighted Sobolev hypothesis follows with every smaller decay rate $q < \tau$. Indeed, for $|\alpha| \le 2$,

$$\begin{aligned} \|\partial^\alpha(g-\delta)\|^p_{L^p_{-q-|\alpha|}} &\le C \int_R^\infty r^{p(q+|\alpha|)} r^{-p(\tau+|\alpha|)} \frac{dr}{r} \\ &= C \int_R^\infty r^{-p(\tau-q)-1}\, dr < \infty. \end{aligned}$$

Thus a C^2 AF end of order $\tau > \frac{n-2}{2}$ gives the Sobolev assumption needed below after choosing q with $\frac{n-2}{2} < q < \tau$. Many analytic statements of the density theorem take this Sobolev condition as the definition of asymptotic flatness.

Remark 6.4.5 (Attribution of the compactification step). There are two different ideas which should not be conflated. The original Schoen–Yau proof of the positive mass theorem uses a noncompact minimal hypersurface produced by a limiting Plateau argument. The more topological shortcut described below—cutting off a negative-mass end, compactifying it to a torus, and then contradicting torus rigidity—is usually attributed to Lohkamp's compactification observation; see [Loh99]. Modern accounts phrase this as follows: using Lohkamp's idea, one can reduce the Riemannian positive mass theorem to the impossibility of $R > 0$ on $\mathbb{T}^n \# X^n$; see the discussion in [LUY24].

Theorem 6.4.6 (Standard deformation to a negative harmonically flat end)

Let (M^n, g) be complete, one-ended, and asymptotically flat, with $R_g \ge 0$ and negative ADM mass $m < 0$. Assume the usual decay for which the ADM mass and the weighted elliptic theory are valid, including $R_g \in L^1$; for instance one may work with the Sobolev AF condition in Definition 6.4.3. Then, for every sufficiently small $\varepsilon > 0$, there is a complete asymptotically flat metric g_ε with the following properties:

$$R_{g_\varepsilon} \ge 0, \qquad |m_{\mathrm{ADM}}(g_\varepsilon) - m| < \varepsilon,$$

and on some exterior coordinate region $\{r \ge R_\varepsilon\}$,

$$g_\varepsilon = u_\varepsilon^{\frac{4}{n-2}}\delta, \qquad \Delta^\delta u_\varepsilon = 0, \qquad u_\varepsilon = 1 + a_\varepsilon r^{2-n} + O_\infty(r^{1-n}).$$

Moreover

$$m_{\mathrm{ADM}}(g_\varepsilon) = 2a_\varepsilon.$$

Thus, if $\varepsilon < -m/2$, then $a_\varepsilon < 0$. In the positive-mass contradiction argument, we may therefore replace the original metric by one which is conformally flat and scalar-flat near infinity, with negative harmonic mass coefficient. ♡

Here the notation in the conclusion is as follows. The function u_ε is a positive harmonic function on the exterior Euclidean coordinate region, and a_ε is the coefficient of its leading r^{2-n} term. The notation $O_\infty(r^\beta)$ is a shorthand for "big-O with all derivatives". More precisely, if $E = O_\infty(r^\beta)$, then for every multi-index α there is a constant C_α such that, in the chosen exterior coordinates,

$$|\partial^\alpha E| \le C_\alpha r^{\beta-|\alpha|} \qquad \text{for } r \text{ large}.$$

Thus $O_\infty(r^{1-n})$ means not only that the error itself is $O(r^{1-n})$, but also that every coordinate derivative has the corresponding differentiated decay; for example

$$\partial^\alpha O_\infty(r^{1-n}) = O(r^{1-n-|\alpha|}).$$

This is stronger than ordinary $O(r^{1-n})$ notation and is the form needed when differentiating the expansion in the ADM mass or scalar-curvature computations. The equality $m_{\mathrm{ADM}}(g_\varepsilon) = 2a_\varepsilon$ is the standard ADM normalization for a conformally flat end $u_\varepsilon^{4/(n-2)}\delta$.

Proof This is the standard density step in the positive mass theorem; see [Bar86, LLU23]. We spell out the mechanism because it is the analytic part which precedes Lohkamp's compactification.

Choose a large radial cut-off χ_λ on the asymptotically flat end, with $\chi_\lambda = 1$ for $r \le \lambda$ and $\chi_\lambda = 0$ for $r \ge 2\lambda$. Let $g_{\mathrm{Euc}} = \delta$ be the Euclidean metric in the AF coordinates and set

$$g_\lambda = \chi_\lambda g + (1-\chi_\lambda)g_{\mathrm{Euc}}$$

on the end, extending it by g on the compact part. Thus $g_\lambda = g$ on a large compact set and $g_\lambda = \delta$ for $r \ge 2\lambda$. The only scalar curvature error is in the annulus $\{\lambda \le r \le 2\lambda\}$. Define

$$V_\lambda := R_{g_\lambda} - \chi_\lambda R_g.$$

The basic estimate is

$$\|V_\lambda\|_{L^p_{-q'-2}} + \|V_\lambda\|_{L^{n/2}} + \|V_\lambda\|_{L^{2n/(n+2)}} \to 0 \tag{6.4.1}$$

for every $q' < q$ with $\frac{n-2}{2} < q' < n-2$. The point is that the linear part of the scalar curvature at the Euclidean metric is

$$DR_\delta(k) = \partial_i\partial_j k_{ij} - \partial_j\partial_j k_{ii}.$$

When $k_\lambda = \chi_\lambda(g-\delta)$, the terms with two derivatives on $g-\delta$ are exactly $\chi_\lambda DR_\delta(g-\delta)$; the remaining terms contain at least one derivative of χ_λ and are supported in the annulus. Since $|\nabla^k\chi_\lambda| \le C\lambda^{-k}$ there, and $g-\delta \in W^{2,p}_{-q}$, these commutator terms go to zero in the weighted norms above. The nonlinear terms in R_{g_λ} contain either $(g_\lambda-\delta)\partial^2 g_\lambda$ or $(\partial g_\lambda)^2$, and have the same decay. This proves (6.4.1).

For λ large, solve the conformal correction equation

$$-c_n\Delta^{g_\lambda}w_\lambda + V_\lambda w_\lambda = 0, \qquad w_\lambda \to 1 \quad \text{on the chosen end,} \tag{6.4.2}$$

where $c_n = \frac{4(n-1)}{n-2}$. The weighted Fredholm estimate for the Laplacian on an asymptotically flat end, together with the smallness of V_λ in (6.4.1), gives a unique solution with

$$w_\lambda - 1 \in W^{2,p}_{-q'}, \qquad \|w_\lambda - 1\|_{W^{2,p}_{-q'}} \to 0.$$

Equivalently, writing $w_\lambda = 1+\eta_\lambda$, the equation is

$$\left(-c_n\Delta^{g_\lambda} + V_\lambda\right)\eta_\lambda = -V_\lambda.$$

The operator on the left is a small perturbation of the AF Laplacian $-c_n\Delta^{g_\lambda} : W^{2,p}_{-q'} \to L^p_{-q'-2}$, which is an isomorphism for $0 < q' < n-2$. Thus a Neumann-series/Fredholm argument solves for η_λ and gives the norm estimate above. This is the only analytic input in the deformation step. Since $p > n$, weighted Sobolev embedding gives $\|w_\lambda - 1\|_{C^0} \to 0$; after increasing λ we therefore have $\frac{1}{2} \le w_\lambda \le 2$. Define

$$\tilde g_\lambda := w_\lambda^{\frac{4}{n-2}} g_\lambda.$$

The conformal scalar curvature formula gives

$$\begin{aligned} R_{\tilde g_\lambda} &= w_\lambda^{-\frac{n+2}{n-2}}\left(-c_n\Delta^{g_\lambda}w_\lambda + R_{g_\lambda}w_\lambda\right) \\ &= w_\lambda^{-\frac{n+2}{n-2}}(R_{g_\lambda} - V_\lambda)w_\lambda = \chi_\lambda R_g\, w_\lambda^{-\frac{4}{n-2}} \ge 0. \end{aligned}$$

Thus the scalar curvature sign is preserved. Since w_λ stays uniformly bounded above and below and g_λ is uniformly equivalent to g, the new metric is complete.

On the region $r \geq 2\lambda$, we have $g_\lambda = \delta$ and $V_\lambda = 0$, so (6.4.2) becomes

$$\Delta^\delta w_\lambda = 0.$$

Therefore $\tilde{g}_\lambda$ is harmonically flat there:

$$\tilde{g}_\lambda = w_\lambda^{\frac{4}{n-2}} \delta, \qquad w_\lambda = 1 + A_\lambda r^{2-n} + O_\infty(r^{1-n}).$$

Here the "conformal mass formula" is the following elementary consequence of the ADM boundary integral. If an AF metric g_0 has mass $m_{\mathrm{ADM}}(g_0)$ and

$$\hat{g} = u^{\frac{4}{n-2}} g_0, \qquad u = 1 + Ar^{2-n} + O_\infty(r^{1-n}),$$

then

$$m_{\mathrm{ADM}}(\hat{g}) = m_{\mathrm{ADM}}(g_0) + 2A. \tag{6.4.3}$$

Indeed, in the flat coordinates of the end, $u^{4/(n-2)} = 1 + \frac{4A}{n-2} r^{2-n} + O_\infty(r^{1-n})$, so the extra contribution to the ADM integrand is

$$\partial_j\big((u^{4/(n-2)} - 1)\delta_{ij}\big) - \partial_i\big((u^{4/(n-2)} - 1)\delta_{jj}\big) = -(n-1)\partial_i(u^{4/(n-2)}),$$

and the boundary integral gives $2A$.

Applying (6.4.3) with $g_0 = g_\lambda$ gives

$$m_{\mathrm{ADM}}(\tilde{g}_\lambda) - m_{\mathrm{ADM}}(g_\lambda) = 2A_\lambda.$$

The equation for w_λ then identifies this coefficient. Integrating

$$-c_n \Delta^{g_\lambda} w_\lambda + V_\lambda w_\lambda = 0$$

over a large coordinate ball and letting the radius tend to infinity, using that $g_\lambda = \delta$ near infinity, gives

$$0 = -c_n \lim_{r\to\infty} \int_{S_r} \partial_\nu w_\lambda \, d\sigma + \int_M V_\lambda w_\lambda \, d\mu_{g_\lambda}.$$

Since $\partial_\nu w_\lambda = -(n-2)A_\lambda r^{1-n} + O(r^{-n})$ and $c_n = \frac{4(n-1)}{n-2}$, this becomes

$$2A_\lambda = -\frac{1}{2(n-1)\omega_{n-1}} \int_M V_\lambda w_\lambda \, d\mu_{g_\lambda}.$$

Hence

$$m_{\mathrm{ADM}}(\tilde{g}_\lambda) - m_{\mathrm{ADM}}(g_\lambda) = -\frac{1}{2(n-1)\omega_{n-1}} \int_M V_\lambda w_\lambda \, d\mu_{g_\lambda},$$

Here g_λ is exactly Euclidean at infinity, so $m_{\mathrm{ADM}}(g_\lambda) = 0$; the integral above is precisely what produces the new harmonic mass coefficient.

To see that this new mass is close to the original mass, one uses the standard mass-continuity lemma in the density theorem. The construction gives $\tilde{g}_\lambda - g \to 0$ in $W^{2,p}_{-q'}$ for every $\frac{n-2}{2} < q' < n-2$, and the scalar curvatures also converge in L^1:

$$R_{\tilde{g}_\lambda} - R_g = \chi_\lambda R_g \big(w_\lambda^{-\frac{4}{n-2}} - 1\big) + (\chi_\lambda - 1)R_g.$$

The first term goes to zero because $w_\lambda \to 1$ uniformly and R_g is integrable on the AF end; the second goes to zero because its support escapes to infinity. The ADM boundary integral is continuous under this $W^{2,p}_{-q'}$ convergence together with L^1 scalar-curvature convergence: this is the usual Bartnik density mass lemma, obtained by writing the mass integrand as the Euclidean linearization of scalar curvature plus quadratic terms, whose tails are integrable when $q' > \frac{n-2}{2}$. Hence

$$m_{\mathrm{ADM}}(\tilde{g}_\lambda) \to m_{\mathrm{ADM}}(g) = m.$$

Choosing λ large, the mass of $\tilde{g}_\lambda$ therefore differs from m by less than ε.

It remains only to identify the coefficient A_λ with the ADM mass. Set $v = w_\lambda^{4/(n-2)}$, so $\tilde{g}_{\lambda,ij} = v\delta_{ij}$ near infinity. Since

$$v = 1 + \frac{4A_\lambda}{n-2} r^{2-n} + O(r^{1-n}), \qquad \partial_\nu v = -4A_\lambda r^{1-n} + o(r^{1-n}),$$

we compute

$$\partial_j \tilde{g}_{\lambda,ij} - \partial_i \tilde{g}_{\lambda,jj} = -(n-1)\partial_i v.$$

Substituting in the ADM formula yields

$$m_{\mathrm{ADM}}(\tilde{g}_\lambda) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} 4A_\lambda (n-1) r^{1-n}\, d\sigma = 2A_\lambda.$$

Taking $g_\varepsilon = \tilde{g}_\lambda$ and $a_\varepsilon = A_\lambda$ proves the theorem. ♦

Lemma 6.4.7 (Lohkamp cut-off on a negative harmonically flat end)

Assume that on an exterior coordinate region $\{r \geq R_0\}$ *the metric is*

$$g = u^{\frac{4}{n-2}}\delta, \qquad R_g \geq 0, \qquad a < 0,$$

$$u = 1 + ar^{2-n} + o(r^{2-n}), \qquad \nabla u = -(n-2)ar^{1-n}\partial_r + o(r^{1-n}).$$

Then, after increasing R_0 *if necessary, one can replace* u *outside a large compact set by a smooth positive function* $\bar{u}$ *such that*

$$\bar{u} = u \quad \text{near the compact core}, \qquad \bar{u} \equiv c > 0 \quad \text{near infinity}, \qquad \Delta^\delta \bar{u} \leq 0,$$

with strict inequality somewhere. Therefore

$$\bar{g} = \bar{u}^{\frac{4}{n-2}}\delta$$

has nonnegative scalar curvature on the end, has positive scalar curvature somewhere in the transition region, and is exactly flat near infinity. Since $\bar{u} = u$ *on a full neighborhood of the inner matching sphere, this replacement glues smoothly to the original metric on the compact part.* ♠

Proof The conformal scalar curvature formula on the flat background gives

$$R_g = -c_n u^{-\frac{n+2}{n-2}}\Delta^\delta u, \qquad c_n = \frac{4(n-1)}{n-2}.$$

Since $R_g \geq 0$, the conformal factor is superharmonic:

$$\Delta^\delta u \leq 0.$$

The negative coefficient $a < 0$ says that u approaches 1 from below. More precisely, for r sufficiently large,

$$\partial_r u = -(n-2)ar^{1-n} + o(r^{1-n}) > 0.$$

After enlarging the compact core, choose $\varepsilon > 0$ and radii $R_1 < R_2$ so large that

$$u < 1 - 3\varepsilon \quad \text{on a collar of } S_{R_1}, \qquad u > 1 - \varepsilon \quad \text{for } r \geq R_2,$$

and such that $\partial_r u > 0$ throughout $R_1 \leq r \leq R_2$. Thus the transition set

$$1 - 3\varepsilon < u < 1 - \varepsilon$$

is contained in a compact annulus in the end, and $|\nabla u| \neq 0$ somewhere inside this transition set.

Choose a smooth function $\Psi : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\Psi(t) = t \quad \text{for } t \leq 1 - 3\varepsilon, \qquad \Psi(t) = c \quad \text{for } t \geq 1 - \varepsilon,$$

where $c > 0$, and

$$0 \leq \Psi' \leq 1, \qquad \Psi'' \leq 0,$$

with $\Psi'' < 0$ somewhere in $(1-3\varepsilon, 1-\varepsilon)$. Such a function is obtained by choosing a smooth nonincreasing function $\theta : [1-3\varepsilon, 1-\varepsilon] \to [0,1]$ which equals 1 near the left endpoint and 0 near the right endpoint, and then setting

$$\Psi(t) = 1 - 3\varepsilon + \int_{1-3\varepsilon}^{t} \theta(s)\, ds$$

on the transition interval, with the two constant/identity extensions above.

Now set

$$\bar{u} := \Psi(u)$$

on the end, and keep $\bar{u} = u$ on the compact core. This is smooth across the inner matching region because $\Psi(t) = t$ wherever $u \leq 1 - 3\varepsilon$. It is positive because $u > 0$ and $c > 0$, and it is constant equal to c near infinity because Ψ is constant for $t \geq 1 - \varepsilon$.

By the chain rule,

$$\Delta^\delta \bar{u} = \Psi''(u)|\nabla u|^2 + \Psi'(u)\Delta^\delta u \leq 0.$$

The inequality is strict somewhere in the transition annulus: there $|\nabla u| \neq 0$ at some point where $\Psi''(u) < 0$. This proves the desired superharmonic cut-off. Notice that no spherical symmetry is used here; the only inputs are conformal flatness, superharmonicity, and the negative mass asymptotic.

Finally use the conformal scalar curvature formula with flat background metric. For any positive function ϕ,

$$R_{\phi^{4/(n-2)}\delta} = -c_n \phi^{-\frac{n+2}{n-2}} \Delta^\delta \phi, \qquad c_n = \frac{4(n-1)}{n-2}.$$

Applying this to $\phi = \bar{u}$ gives

$$R_{\bar{g}} = -c_n \bar{u}^{-\frac{n+2}{n-2}} \Delta^\delta \bar{u} \geq 0,$$

and it is positive somewhere. Since $\bar{u}$ is constant near infinity, $\bar{g}$ is flat there. ◆

Proposition 6.4.8 (Negative mass produces a PSC torus connected sum)

If a one-ended asymptotically flat manifold (M^n, g), $3 \leq n \leq 7$, has nonnegative scalar curvature and negative ADM mass, then some closed manifold of the form

$$\mathbb{T}^n \# Y$$

admits a positive-scalar-curvature metric. ♠

Proof First apply Theorem 6.4.6, choosing the deformation so that the mass remains negative. Then apply Lemma 6.4.7 to the harmonically flat end. We obtain a new complete metric, still denoted by g, with the following properties: $R_g \geq 0$, $R_g > 0$ somewhere in a compact annulus in the chosen end, and on $\{r \geq R_2\}$ the metric is

$$g = c^{\frac{4}{n-2}} \delta$$

for a constant $c > 0$.

Choose a coordinate radius R with $R_0 < R < R_2$, and let $K_R := M \setminus \{r > R\}$ be the compact manifold obtained by cutting the chosen end at the coordinate sphere $S_R = \partial B_R$. Then choose $L > R_2$ and let

$$Q_L = [-L, L]^n \subset \mathbb{R}^n.$$

Since every point of ∂Q_L has Euclidean radius at least $L > R_2$, a whole collar of ∂Q_L lies in the exactly flat region. Remove from M the part of the chosen end outside Q_L. Equivalently, the remaining compact manifold with corners is

$$W = K_R \cup_{S_R} (Q_L \setminus B_R),$$

where $Q_L \setminus B_R$ is read in the asymptotic coordinate chart. The outer boundary of this fundamental domain is the boundary of the cube Q_L.

Now identify opposite faces of ∂Q_L by the translations

$$(x^1, \dots, x^i = L, \dots, x^n) \sim (x^1, \dots, x^i = -L, \dots, x^n), \qquad i = 1, \dots, n.$$

These translations are isometries for the constant flat metric $c^{4/(n-2)}\delta$. Hence the metric descends smoothly across the identified faces. There is no corner singularity: the cube is only a fundamental domain for the standard smooth quotient $Q_L/\sim = \mathbb{T}^n$, and the metric is the translation-invariant flat metric in a neighborhood of the boundary faces.

Let X be the closed quotient. We next identify its topology. Form the closed manifold

$$Y := K_R \cup_{S_R} B_R.$$

On the other hand, after the opposite faces of Q_L are identified, Q_L becomes $\mathbb{T}^n$, and the image of the interior ball B_R is an embedded ball in this torus. Therefore

$$X = K_R \cup_{S_R} \big((Q_L \setminus B_R)/\sim\big) \simeq Y \# \mathbb{T}^n.$$

This is the promised torus connected sum.

The scalar curvature statement is local, so it survives the quotient. Thus the induced metric on X has $R \geq 0$ everywhere and $R > 0$ somewhere. By Lemma 6.1.2, the closed manifold X admits a metric with $R > 0$ everywhere. Since $X \simeq \mathbb{T}^n \# Y$, this proves the proposition. ◆

Proof of the nonnegativity statement in Theorem 6.4.2 Suppose, to the contrary, that the mass is negative. By Proposition 6.4.8, some $\mathbb{T}^n \# Y$ admits a positive-scalar-curvature metric. This contradicts Corollary 6.3.2. Hence the ADM mass is nonnegative. ◆

Remark 6.4.9 (Rigidity). The equality case is less topological but fits the same philosophy. If an asymptotically flat manifold with $R_g \geq 0$ has zero mass and is not Euclidean, one uses a conformal/deformation argument to produce a new asymptotically flat metric with $R \geq 0$ and strictly negative mass. This contradicts the nonnegativity just proved. In the original Schoen–Yau proof this is combined with the strong maximum principle and the regularity theory for the minimizing hypersurfaces.

Remark 6.4.10 (Summary of the positive mass argument). The proof has three moving parts.

1. Stability plus the Gauss equation gives
$$\int_\Sigma 2|\nabla\phi|^2 + R_\Sigma \phi^2 \geq \int_\Sigma (R_N + |A|^2)\phi^2.$$
This is the whole reason positive scalar curvature descends to a stable minimal hypersurface.
2. A nonzero-degree map to a torus supplies cohomology classes whose cup product remains nonzero after passing to a Poincare-dual minimizing hypersurface.
3. A negative-mass end can be flattened to make a closed positive-scalar-curvature metric on $\mathbb{T}^n \# Y$, contradicting Geroch.

6.5 The Horowitz–Myers Conjecture

The Horowitz–Myers conjecture [HM98] is a positive mass statement for spaces with negative cosmological constant. The point of this section is to explain the Riemannian version proved by Brendle–Hung [BH24], and then to outline how their systolic inequality proves it.

6.5.1 From AdS energy to a Riemannian inequality

Recall first the analogy with the positive mass theorem. For asymptotically flat initial data, Euclidean space is the reference geometry and the expected lower bound for the ADM mass is 0. With a negative cosmological constant, the natural reference geometry is anti-de Sitter space. In spacetime language the Einstein equation is

$$\mathrm{Ric}_{\mathbf{g}} - \frac{1}{2}R_{\mathbf{g}}\mathbf{g} + \Lambda\mathbf{g} = 8\pi T.$$

For AdS geometry one takes $\Lambda < 0$; in our normalization,

$$\Lambda = -\frac{n(n-1)}{2}.$$

On a time-symmetric spacelike slice, the second fundamental form vanishes and the Hamiltonian constraint becomes a scalar-curvature condition. In vacuum it is

$$R_g = -n(n-1),$$

and under the dominant energy condition it becomes

$$R_g \geq -n(n-1).$$

For asymptotically hyperbolic data with spherical conformal infinity, the ground state is pure AdS, whose time-symmetric slice is hyperbolic space. The Horowitz–Myers phenomenon begins when the conformal infinity is toroidal. In that case the product hyperbolic end is not the expected lowest-energy geometry. One circle direction at infinity can fill in smoothly in the interior, producing the AdS soliton on $\mathbb{R}^2 \times T^{n-2}$. This metric has negative mass relative to the product hyperbolic end. The conjecture says that, among metrics with the same toroidal asymptotics and $R_g \geq -n(n-1)$, the AdS soliton has the least possible mass.

6.5.2 The Brendle–Hung mass inequality

Let γ be a flat metric on $S^1 \times T^{n-2}$. The model end is

$$\bar{g} = r^{-2}dr^2 + r^2\gamma$$

on $(r_0, \infty) \times S^1 \times T^{n-2}$. An asymptotically Horowitz–Myers end has an expansion

$$g = \bar{g} + r^{2-n}Q + o(r^{2-n}),$$

where Q is a symmetric 2-tensor on $S^1 \times T^{n-2}$. The quantity

$$\int_{S^1 \times T^{n-2}} n\,\mathrm{tr}_\gamma Q\, dV_\gamma$$

is the mass term in the normalization used here.

There is also a systolic quantity built into the toroidal end. Let $\xi : S^1 \times T^{n-2} \to S^1$ be the circle projection and let Ξ be the pullback of the volume form on S^1. Define

$$\sigma = \inf\left\{\mathrm{length}_\gamma(\alpha) : \alpha \subset S^1 \times T^{n-2} \text{ closed and } \int_\alpha \Xi \neq 0\right\}.$$

Thus σ is the length of the shortest loop which winds nontrivially in the distinguished S^1-direction.

Theorem 6.5.1 (Horowitz–Myers mass inequality, Brendle–Hung)

Let $3 \leq n \leq 7$, and let γ be a flat metric on $S^1 \times T^{n-2}$. Let

$$\bar{g} = r^{-2}dr^2 + r^2\gamma$$

on $(r_0, \infty) \times S^1 \times T^{n-2}$*, and let* Q *be a symmetric* 2*-tensor on* $S^1 \times T^{n-2}$*. Suppose* (N, g_N) *is a smooth* n*-manifold such that:*

(1) $N \setminus E \cong (r_0, \infty) \times S^1 \times T^{n-2}$ *for some compact set* $E \subset N$*;*

(2) the T^{n-2}*-projection on the end extends smoothly to a map* $N \to T^{n-2}$*;*

(3) on the end,

$$|g_N - \bar{g} - r^{2-n} Q|_{\bar{g}} = o(r^{-n}), \qquad |\bar{D}(g_N - \bar{g} - r^{2-n} Q)|_{\bar{g}} = o(r^{-n});$$

(4) $R_{g_N} \geq -n(n-1)$*.*

Then

$$\int_{S^1 \times T^{n-2}} n \, \mathrm{tr}_\gamma(Q) \, dV_\gamma \geq - \int_{S^1 \times T^{n-2}} \left(\frac{4\pi}{n\sigma} \right)^n dV_\gamma.$$

♡

The left-hand side is the Horowitz–Myers mass term. The right-hand side is the mass of the corresponding AdS soliton. The rigidity statement, proved by Brendle–Hung in a subsequent work [BH25], says that equality forces the metric to be locally isometric to a Horowitz–Myers metric.

6.5.3 The systolic boundary inequality

The mass inequality is proved by cutting off the end and applying a sharp boundary inequality to the resulting compact manifold. We state that inequality in the compact form in which it is used.

Let M be a compact, connected, orientable n-manifold with nonempty boundary. Suppose

$$\xi : \partial M \to S^1, \qquad \theta = (\theta_1, \ldots, \theta_{n-2}) : \partial M \to T^{n-2}$$

are smooth maps such that $(\xi, \theta) : \partial M \to S^1 \times T^{n-2}$ has nonzero degree. Let $\Xi = \xi^*(d\theta_{S^1})$, where $d\theta_{S^1}$ is the volume form of the circle. Similarly, let $\Theta_j = \theta_j^*(d\theta_j)$ denote the pulled-back volume forms from the circle factors of T^{n-2}. Define σ to be the shortest length of a closed curve $\alpha \subset \partial M$ with $\int_\alpha \Xi \neq 0$. Let $H_{\partial M}$ be the mean curvature of ∂M with respect to the outward unit normal η.

Theorem 6.5.2 (Brendle–Hung systolic boundary inequality)

Let $3 \leq n \leq 7$*, let* $\beta > n$*, and let* $\varphi \in C^\infty(M)$*. If*

$$-2\Delta^M \varphi - \frac{\beta - n + 1}{\beta - n} |\nabla^M \varphi|^2 + R_M + \beta(\beta - 1) \geq 0,$$

then

$$2\sigma^\beta \inf_{\partial M} \left(\langle \nabla^M \varphi, \eta \rangle + H_{\partial M} - (\beta - 1) \right) \leq \left(\frac{4\pi}{\beta} \right)^\beta.$$

♡

For the mass theorem, one only needs the scalar-curvature consequence obtained by taking $\varphi \equiv 0$ and letting $\beta \to n$.

Corollary 6.5.3 (Scalar-curvature boundary inequality)

With the same topological notation, if $R_M \geq -n(n-1)$*, then*

$$2\sigma^n \inf_{\partial M} \left(H_{\partial M} - (n-1) \right) \leq \left(\frac{4\pi}{n} \right)^n.$$

♠

6.5.4 The AdS soliton and sharpness

The sharp example is the Horowitz–Myers, or AdS soliton, metric. Geometrically one should picture the S^1-factor at infinity as a polar-angle direction which collapses smoothly in the interior. Thus the underlying manifold is $\mathbb{R}^2 \times T^{n-2}$, while the conformal infinity is $S^1 \times T^{n-2}$.

For simplicity, assume that (S^1, g_{S^1}) has length $4\pi/n$. On $(1,\infty) \times S^1 \times T^{n-2}$, set

$$g = \frac{1}{\rho^2(1-\rho^{-n})}d\rho^2 + \rho^2(1-\rho^{-n})g_{S^1} + \rho^2 g_{T^{n-2}}.$$

With the substitution

$$\rho = \left(\cosh \frac{n\tilde{\rho}}{2}\right)^{2/n},$$

this becomes

$$g = d\tilde{\rho}^2 + \left(\cosh \frac{n\tilde{\rho}}{2}\right)^{4/n}\left[\left(\tanh \frac{n\tilde{\rho}}{2}\right)^2 g_{S^1} + g_{T^{n-2}}\right].$$

Near $\tilde{\rho} = 0$, the first two directions look like

$$d\tilde{\rho}^2 + \left(\frac{n\tilde{\rho}}{2}\right)^2 g_{S^1}.$$

The choice $\mathrm{length}(S^1) = 4\pi/n$ is exactly the no-cone-angle condition, so the metric extends smoothly across the collapsed circle. The resulting metric has scalar curvature

$$R_g = -n(n-1).$$

To compare it with the asymptotic model, define r by

$$\rho^{n/2} = r^{n/2}\left(1 + \frac{1}{4}r^{-n}\right).$$

Then

$$\begin{aligned} g &= r^{-2}dr^2 + r^2\left(1+\frac{1}{4}r^{-n}\right)^{4/n-2}\left(1-\frac{1}{4}r^{-n}\right)^2 g_{S^1} + r^2\left(1+\frac{1}{4}r^{-n}\right)^{4/n} g_{T^{n-2}} \\ &= r^{-2}dr^2 + r^2\left(1-\frac{n-1}{n}r^{-n}\right)g_{S^1} + r^2\left(1+\frac{1}{n}r^{-n}\right)g_{T^{n-2}} + o(r^{-n}). \end{aligned}$$

Thus the tensor Q is

$$Q = -\frac{n-1}{n}g_{S^1} + \frac{1}{n}g_{T^{n-2}},$$

and

$$\operatorname{tr}_\gamma Q = -\frac{1}{n}.$$

Since $\sigma = 4\pi/n$, the right-hand side in Theorem 6.5.1 is exactly the mass of this metric. Hence the inequality is sharp.

6.5.5 How the Brendle–Hung proof works

We keep the proof in three parts. The first part converts the boundary inequality into the mass inequality. The second part proves the boundary inequality by dimension reduction. The final part is the two-dimensional endpoint, where the sharp constant is computed.

Step 1: from the boundary inequality to the mass inequality.

The goal is to prove Theorem 6.5.1:

$$\int_{S^1\times T^{n-2}} n\,\mathrm{tr}_\gamma\, Q\, dV_\gamma \geq -\int_{S^1\times T^{n-2}} \left(\frac{4\pi}{n\sigma}\right)^n dV_\gamma.$$

The compact input is Corollary 6.5.3. Thus we need to cut off the end of N in such a way that the boundary mean curvature sees the mass aspect.

Choose u and a constant μ on $S^1 \times T^{n-2}$ by

$$\Delta^\gamma u + \frac{n}{2}\,\mathrm{tr}_\gamma\, Q + \mu = 0, \qquad \int_{S^1\times T^{n-2}} u\, dV_\gamma = 0.$$

Equivalently,

$$\int_{S^1\times T^{n-2}} (n\,\mathrm{tr}_\gamma\, Q + 2\mu)\, dV_\gamma = 0.$$

For large $\widehat{r}$, cut off the end by the graph

$$\widehat{N} = N \cap \{r \leq \widehat{r} + \widehat{r}^{3-n}\widehat{u}\}, \qquad \widehat{u} = u \circ \pi.$$

The reason for using this graph, rather than the coordinate torus $\{r = \widehat{r}\}$, is that the graph makes the first nontrivial term in the mean curvature constant. The two estimates one needs are

$$D^2\widehat{u} = D^2_\gamma u - r^{-1}(dr \otimes d\widehat{u} + d\widehat{u} \otimes dr) + O(r^{-n-1})$$

and

$$D^2 r = rg - \frac{n}{2} r^{3-n} Q + o(r^{1-n}).$$

Substituting these into the level-set formula

$$H_{\partial\widehat{N}} = \frac{\mathrm{tr}_{\partial\widehat{N}}\, D^2(r - \widehat{r}^{3-n}\widehat{u})}{|D(r - \widehat{r}^{3-n}\widehat{u})|}$$

gives

$$H_{\partial\widehat{N}} = (n-1) + \widehat{r}^{-n}\mu + o(\widehat{r}^{-n}).$$

Therefore Corollary 6.5.3 applied to $\widehat{N}$ gives

$$2\widehat{\sigma}^n\widehat{r}^{-n}\mu \leq \left(\frac{4\pi}{n}\right)^n + o(1),$$

where $\widehat{\sigma}$ is the boundary systole on $\partial\widehat{N}$. Since

$$\widehat{r}^{-2} g_{\partial\widehat{N}} \to \gamma, \qquad \frac{\widehat{\sigma}}{\widehat{r}} \to \sigma,$$

we pass to the limit and recover the mass inequality. Equivalently, one may argue by contradiction: if the mass integral were too negative, then for some $\varepsilon > 0$,

$$2(1-\varepsilon)^{n+1}\sigma^n\mu \geq \left(\frac{4\pi}{n}\right)^n,$$

whereas the boundary inequality gives

$$2(1-\varepsilon)\widehat{\sigma}^n\widehat{r}^{-n}\mu \leq \left(\frac{4\pi}{n}\right)^n$$

for large $\widehat{r}$. Hence $\widehat{\sigma}/\widehat{r} \leq (1-\varepsilon)\sigma$, contradicting $\widehat{\sigma}/\widehat{r} \to \sigma$.

Step 2: dimensional reduction for the systolic inequality.

The goal is now Theorem 6.5.2. The proof reduces the n-dimensional boundary inequality to a two-dimensional one by repeatedly taking free-boundary hypersurfaces which preserve the relevant cohomological information.

The inductive object is a pair (Σ^k, φ_k), where Σ^k has boundary and carries the nontrivial topological information coming from $\Xi, \Theta_1, \dots, \Theta_{k-2}$. The pair is required to satisfy

$$E_k(\varphi_k, g_k) := -2\Delta^{\Sigma^k}\varphi_k - \frac{\beta - k + 1}{\beta - k}|\nabla^{\Sigma^k}\varphi_k|^2 + R_{\Sigma^k} + \beta(\beta - 1) \geq 0,$$

and the boundary term is

$$B_k := \partial_\eta \varphi_k + H_{\partial\Sigma^k}.$$

Starting with $\Sigma^n = M$ and $\varphi_n = \varphi$, the reduction step is:

Reduction proposition.

Assume $E_k(\varphi_k, g_k) \geq 0$ and $B_k \geq 0$. Then there is a compact free boundary hypersurface $\Sigma^{k-1} \subset \Sigma^k$, stable for the weighted area functional

$$\mathcal{A}_k(S) = \int_S e^{\varphi_k}\, dA_{g_k},$$

and a function φ_{k-1} on Σ^{k-1}, such that

$$E_{k-1}(\varphi_{k-1}, g_{k-1}) \geq 0$$

and

$$\partial_\eta \varphi_{k-1} + H_{\partial\Sigma^{k-1}} = \partial_\eta \varphi_k + H_{\partial\Sigma^k}.$$

The hypersurface is chosen in the homology class detected by the remaining forms, so the relevant systole cannot decrease in the direction needed for the final inequality.

Let us indicate why the differential inequality is preserved. Write $\bar\Sigma = \Sigma^k$, $\widetilde\Sigma = \Sigma^{k-1}$, and $\bar\varphi = \varphi_k|_{\widetilde\Sigma}$. All geometric quantities below are computed in the original metric g_k. If $\widetilde\nu$ is the unit normal of $\widetilde\Sigma \subset \bar\Sigma$, then the second variation of $\mathcal{A}_k$ gives, for every smooth test function ζ,

$$\begin{aligned} 0 \leq \int_{\widetilde\Sigma} e^{\bar\varphi}\left(|\widetilde\nabla\zeta|^2 - \left(\overline{\mathrm{Ric}}(\widetilde\nu, \widetilde\nu) + |\widetilde A|^2\right)\zeta^2 + \bar\nabla^2\bar\varphi(\widetilde\nu, \widetilde\nu)\zeta^2\right) \\ - \int_{\partial\widetilde\Sigma} e^{\bar\varphi} A_{\partial\bar\Sigma}(\widetilde\nu, \widetilde\nu)\zeta^2. \end{aligned}$$

Here $\widetilde A$ is the second fundamental form of $\widetilde\Sigma \subset \bar\Sigma$. The usual Gauss equation and the free-boundary relation give

$$|\widetilde A|^2 + \overline{\mathrm{Ric}}(\widetilde\nu, \widetilde\nu) = \frac{1}{2}\left(\bar R - \widetilde R + |\widetilde A|^2 + \widetilde H^2\right) \geq \frac{1}{2}\left(\bar R - \widetilde R + \widetilde H^2\right),$$

and

$$A_{\partial\bar\Sigma}(\widetilde\nu, \widetilde\nu) = H_{\partial\bar\Sigma} - H_{\partial\widetilde\Sigma}.$$

Thus the stability inequality may be weakened to

$$\begin{aligned} 0 \leq \int_{\widetilde\Sigma} e^{\bar\varphi}\left(|\widetilde\nabla\zeta|^2 - \frac{1}{2}(\bar R - \widetilde R + \widetilde H^2)\zeta^2 + \bar\nabla^2\bar\varphi(\widetilde\nu, \widetilde\nu)\zeta^2\right) \\ - \int_{\partial\widetilde\Sigma} e^{\bar\varphi}(H_{\partial\bar\Sigma} - H_{\partial\widetilde\Sigma})\zeta^2. \end{aligned}$$

By the first-eigenfunction argument for this Robin problem, there is a positive function ω on $\widetilde\Sigma$ and a number $\lambda \geq 0$ such that

$$\begin{aligned} -\operatorname{div}^{\widetilde\Sigma}(e^{\bar\varphi}\widetilde\nabla\omega) - \frac{1}{2}(\bar R - \widetilde R + \widetilde H^2)e^{\bar\varphi}\omega + e^{\bar\varphi}\bar\nabla^2\bar\varphi(\widetilde\nu, \widetilde\nu)\omega \\ = \lambda e^{\bar\varphi}\omega \geq 0, \end{aligned}$$

with boundary condition

$$\partial_\eta\omega - (H_{\partial\bar\Sigma} - H_{\partial\widetilde\Sigma})\omega = 0.$$

Define

$$\widetilde{\varphi} = \varphi_{k-1} := \bar{\varphi} + \log \omega.$$

The Schoen–Yau rearrangement of the stability inequality gives

$$\begin{aligned} &-2\widetilde{\Delta}\widetilde{\varphi} + \widetilde{R} - |\widetilde{\nabla}\widetilde{\varphi}|^2 \\ &\qquad \geq -2\bar{\Delta}\bar{\varphi} + \bar{R} - |\bar{\nabla}\bar{\varphi}|^2 + |\widetilde{\nabla}\bar{\varphi} - \widetilde{\nabla}\widetilde{\varphi}|^2. \end{aligned}$$

The only algebraic point is

$$|a-b|^2 + p|a|^2 \geq \frac{p}{1+p}|b|^2.$$

Taking $p = 1/(\beta - k)$ gives

$$|\widetilde{\nabla}\bar{\varphi} - \widetilde{\nabla}\widetilde{\varphi}|^2 + \frac{1}{\beta - k}|\bar{\nabla}\bar{\varphi}|^2 \geq \frac{1}{\beta - k + 1}|\widetilde{\nabla}\widetilde{\varphi}|^2.$$

Therefore

$$\begin{aligned} &-2\widetilde{\Delta}\widetilde{\varphi} + \widetilde{R} - \left(1 + \frac{1}{\beta - k + 1}\right)|\widetilde{\nabla}\widetilde{\varphi}|^2 + \beta(\beta - 1) \\ &\qquad \geq -2\bar{\Delta}\bar{\varphi} + \bar{R} - \left(1 + \frac{1}{\beta - k}\right)|\bar{\nabla}\bar{\varphi}|^2 + \beta(\beta - 1) \geq 0. \end{aligned}$$

This is exactly the inequality $E_{k-1} \geq 0$. The boundary equality follows directly from the Neumann condition for ω:

$$\partial_\eta \widetilde{\varphi} + H_{\partial\Sigma^{k-1}} = \partial_\eta \bar{\varphi} + H_{\partial\Sigma^k}.$$

Iterating the reduction either stops early, in which case the boundary infimum is already nonpositive and the desired inequality is immediate, or reaches a surface (Σ^2, φ_2). In the latter case the boundary terms are the same along the construction and, if σ_2 is the corresponding systole on $\partial\Sigma^2$, then $\sigma \leq \sigma_2$. The two-dimensional estimate therefore gives

$$\begin{aligned} &2\sigma^\beta \inf_{\partial M} \left(\partial_\eta \varphi + H_{\partial M} - (\beta - 1)\right) \\ &\qquad \leq 2\sigma_2^\beta \inf_{\partial\Sigma^2} \left(\partial_\eta \varphi_2 + H_{\partial\Sigma^2} - (\beta - 1)\right) \leq \left(\frac{4\pi}{\beta}\right)^\beta. \end{aligned}$$

Step 3: the two-dimensional endpoint and the monotonicity.

The dimension reduction leaves a sharp inequality on a surface. This is the only place where the numerical constant $(4\pi/\beta)^\beta$ is produced explicitly.

Theorem 6.5.4 (Two-dimensional endpoint)

Let Σ be a compact connected orientable surface with nonempty boundary. Let K be its Gaussian curvature, κ the geodesic curvature of $\partial\Sigma$, and η the outward unit normal. Suppose

$$-2\Delta\psi - \frac{\beta - 1}{\beta - 2}|\nabla\psi|^2 + 2K + \beta(\beta - 1) \geq 0.$$

Then:

(i) If Σ is diffeomorphic to a disk, then

$$2|\partial\Sigma|^\beta \inf_{\partial\Sigma} \left(\partial_\eta \psi + \kappa - (\beta - 1)\right) \leq \left(\frac{4\pi}{\beta}\right)^\beta.$$

(ii) If Σ is not diffeomorphic to a disk, then

$$\inf_{\partial\Sigma} \left(\partial_\eta \psi + \kappa - (\beta - 1)\right) \leq 0.$$

♡

The second case is proved by reducing further to a free-boundary geodesic. The main new estimate is the disk case.

For the disk case define the parallel domains

$$\Omega_s = \{x \in \Sigma : d(x, \partial\Sigma) > s\}, \qquad \gamma_s = \partial\Omega_s, \qquad L(s) = \mathcal{H}^1(\gamma_s),$$

and let $l = \sup\{s : \Omega_s \neq \emptyset\}$. Put

$$F(s) = \tanh \frac{\beta s}{2}, \qquad G(s) = \left(\cosh \frac{\beta s}{2}\right)^{\frac{2(\beta-1)}{\beta}}.$$

For almost every s, the curve γ_s is piecewise smooth. If $\Gamma(s)$ denotes its total curvature, then the comparison geometry gives

$$L'(s) \leq -\Gamma(s).$$

Since the connected components of Ω_s are disks, Gauss–Bonnet gives

$$2\pi \leq \Gamma(s) + \int_{\Omega_s} K.$$

Now define

$$I(s) = 2\pi - (\beta-1)F(l-s)L(s) + \int_{\Omega_s} (\Delta\psi - K).$$

The fundamental monotonicity statement is

$$I'(s) - (\beta-1)F(l-s)I(s) \geq 0 \qquad \text{for a.e. } s \in (0, l).$$

The computation is short enough to record the structure. Differentiate I, use $L'(s) \leq -\Gamma(s)$, use Gauss–Bonnet to replace $\Gamma(s)$, and use the scalar inequality to control the integral over γ_s. The last point is the elementary estimate

$$\frac{\beta-1}{2(\beta-2)}|\nabla\psi|^2 + \frac{(\beta-1)(\beta-2)}{2}F^2(l-s) \geq (\beta-1)F(l-s)|\nabla\psi|.$$

These inequalities combine to give

$$I'(s) \geq (\beta-1)F(l-s)I(s).$$

Since

$$(\log G)'(s) = (\beta-1)F(s),$$

the equivalent formulation is

$$J(s) := G(l-s)I(s) \qquad \Longrightarrow \qquad J'(s) \geq 0.$$

This is the monotonicity one should remember.

Finally $J(0) \leq J(l)$. Using Gauss–Bonnet on the original disk, this gives

$$2\pi \geq G(l)\left(\int_{\partial\Sigma} (\partial_\eta\psi + \kappa) - (\beta-1)F(l)|\partial\Sigma|\right).$$

Let $\sigma = |\partial\Sigma|$ in the disk case. Write

$$a := \inf_{\partial\Sigma} (\partial_\eta\psi + \kappa - (\beta-1)).$$

The preceding inequality controls the boundary average, hence also the infimum:

$$a \leq \frac{1}{\sigma}\int_{\partial\Sigma} (\partial_\eta\psi + \kappa - (\beta-1)) \leq \frac{2\pi}{\sigma G(l)} - (\beta-1)(1-F(l)).$$

Multiplying by $2\sigma^\beta$ gives

$$2\sigma^\beta a \leq \frac{4\pi\sigma^{\beta-1}}{G(l)} - 2(\beta-1)(1-F(l))\sigma^\beta.$$

Since $0 \leq F(l) < 1$, we have

$$1 - F(l) \geq \frac{1}{2}(1 - F^2(l)) = \frac{1}{2}G(l)^{-\frac{\beta}{\beta-1}}.$$

Therefore

$$2\sigma^\beta a \leq \frac{4\pi\sigma^{\beta-1}}{G(l)} - (\beta - 1)\frac{\sigma^\beta}{G(l)^{\frac{\beta}{\beta-1}}}.$$

Now set

$$x = \frac{\sigma}{G(l)^{1/(\beta-1)}}.$$

The right-hand side becomes

$$4\pi x^{\beta-1} - (\beta - 1)x^\beta.$$

This one-variable expression is maximized at $x = 4\pi/\beta$, and its maximum is

$$\left(\frac{4\pi}{\beta}\right)^\beta.$$

This proves Theorem 6.5.4. Step 2 gives Theorem 6.5.2, Corollary 6.5.3 follows by taking $\varphi \equiv 0$ and $\beta \to n$, and Step 1 proves the Horowitz–Myers mass inequality.

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