Curvature Estimates for Stable Minimal Triple Junction Surfaces

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First Part

1. Basic Facts about minimal surfaces
2. Curvature estimate
3. Sketch proof of Schoen, Simon, Yau's result

Minimal surfaces in $\mathbb{R}^3$

Definition

A surface $\Sigma \subset \mathbb{R}^3$ is called a *minimal surface* in $\mathbb{R}^3$ if the mean curvature vector $\vec{H}_\Sigma$ of Σ vanishes identically.

Proposition

$\Sigma \subset \mathbb{R}^3$ is a minimal surface if and only if one of the following equivalent conditions holds

- Σ is a critical point of area function.
- The coordinate function $x_i, i = 1,2,3$ is harmonic on Σ, i.e. $\Delta_\Sigma x_i = 0$.

Why these definitions are equivalent?

- First Variation Formula:

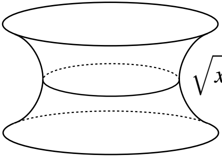
$$\delta\Sigma(X) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Area}(\Sigma_t) = -\int_\Sigma \vec{H}_\Sigma \cdot X \, dA$$

- For any $\Sigma \subset \mathbb{R}^3$ with mean curvature vector $\vec{H}$ and $\vec{x} = (x_1, x_2, x_3)$ be the position vector, we have

$$\Delta_\Sigma \vec{x} = \vec{H}_\Sigma$$

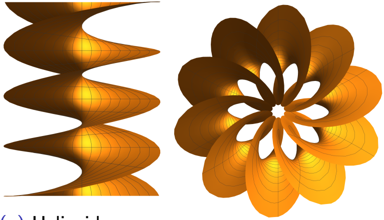
Several examples of minimal surface in $\mathbb{R}^3$

- Standard Catenoid



$$\sqrt{x_1^2 + x_2^2} = \cosh(x_3)$$

- Minimal surface of Helicoid type



(a) Helicoid
$(\rho\cos(\theta), \rho\sin(\theta), \theta)$

(b) Bent Helicoid

- Björling's formula $\mathrm{Re}(c(z) - i\int_0^z n(w) \times c'(w)dw)$.

Second variation formula and stability

Second variation formula

Let Σ be a minimal surface in $\mathbb{R}^3$, then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \mathrm{Area}(\Sigma_t) = \int_\Sigma |\nabla_\Sigma \phi|^2 - |A_\Sigma|^2 \phi^2 \, dA$$

with $\phi = X \cdot \nu$.

Stable minimal surface

We say $\Sigma \subset \mathbb{R}^3$ is a stable minimal surface if $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \mathrm{Area}(\Sigma_t) \geq 0$ for any variation Σ_t of Σ.

Corollary (Stability inequality)

If $\Sigma \subset \mathbb{R}^3$ is stable, then

$$\int_\Sigma |A_\Sigma|^2 \phi^2 \leq \int_\Sigma |\nabla_\Sigma \phi|^2$$

for any Lipschitz function ϕ with compact support.

Some results about curvature estimates for stable minimal surfaces.

- **Heinz (1952)** Curvature estimate for minimal graphs. If $u : D_R \to \mathbb{R}$ satisfies the minimal surface equation, then

$$\sigma^2 \sup_{D_{R-\sigma}} \left| A_{\mathrm{Graph}(u)} \right|^2 \le C$$

- **Schoen-Simon-Yau (1975)** L^p estimate of curvature. Suppose $\Sigma^{n-1} \subset \mathbb{R}^n$ is an orientable stable minimal hypersurface. Then for all $p \in [2, 2 + \sqrt{\frac{2}{n-1}})$ and each nonnegative Lipschitz function ϕ with compact support, we have

$$\int_\Sigma |A_\Sigma|^{2p} \phi^{2p} \le C(n,p) \int_\Sigma |\nabla \phi|^{2p}$$

- **Schoen (1983)** Pointwise curvature estimate for stable minimal surface. If $\Sigma \subset \mathbb{R}^3$ is an orientable stable minimal surface which has no boundary in $B_R(p)$, then for all $0 < \sigma \le R$,

$$\sup_{B_{R-\sigma}(p)} |A_\Sigma|^2 \le \frac{C}{\sigma^2}$$

Generalized Bernstein Theorem

L^p estimate of curvature $\Longrightarrow$ Generalized Bernstein Theorem.

Theorem (Schoen-Simon-Yau)

If $\Sigma^{n-1} \subset \mathbb{R}^n$ is a complete two-sided stable minimal hypersurface with $n \leq 6$, and there exists $C < \infty$ such that

$$\sup_{R>0} \frac{\mathrm{Vol}(B_R \cap \Sigma)}{R^{n-1}} \leq C$$

then Σ is flat.

Theorem (Colding-Minicozzi)

If $\Sigma^2 \subset \mathbb{R}^3$ is stable and two sided, B_r^Σ is simply connected, then $\mathrm{Area}(B_r^\Sigma) \leq \frac{4\pi}{3} r^2$.

Corollary (Bernstein type Theorem for $n = 3$)

If $\Sigma \subset \mathbb{R}^3$ is complete, stable and 2-sided, then it is flat.

Sketch proof of L^p estimates

Theorem (Schoen-Simon-Yau)

Suppose $\Sigma^{n-1} \subset \mathbb{R}^n$ *is a two-sided stable minimal hypersurface. For* $2 \le p < 2 + \sqrt{\frac{2}{n-1}}$ *and* $\phi \in C_c^1(\Sigma)$, *we have*

$$\int_\Sigma |A_\Sigma|^{2p} \phi^{2p} \le C(p) \int_\Sigma |\nabla \phi|^{2p}$$

Key Tools in the proof, Simon's inequality

Suppose $\Sigma^{n-1} \subset \mathbb{R}^n$ is a minimal hypersurface, then

$$|A_\Sigma| \Delta_\Sigma |A_\Sigma| + |A_\Sigma|^4 \ge \frac{2}{n-1} |\nabla_\Sigma |A_\Sigma||^2$$

Sketch proof of L^p estimates

- Change $\phi \to \phi |A|^{p-1}$ in stability inequality ($\int_\Sigma |A|^2 \phi^2 \le \int_\Sigma |\nabla\phi|^2$) to get

$$\int_\Sigma |A|^{2p}\phi^2 \le (p-1)^2 \int_\Sigma \phi^2 |\nabla|A||^2 |A|^{2p-4} + \int |A|^{2p-2}|\nabla\phi|^2 + 2(p-1)\int_\Sigma \phi |A|^{2p-3}\nabla\phi\cdot\nabla|A|.$$

- Multiply Simon's inequality ($\frac{2}{n-1}|\nabla|A||^2 \le |A|\Delta_\Sigma|A| + |A|^4$) by $|A|^{2p-4}\phi^2$ and integrate

$$\frac{2}{n-1}\int_\Sigma |\nabla|A||^2|A|^{2p-4}\phi^2 \le \int_\Sigma |A|^{2p}\phi^2 - 2\phi|A|^{2p-3}\nabla\phi\cdot\nabla|A| - (2p-3)\int_\Sigma \phi^2|A|^{2p-4}|\nabla|A||^2.$$

- Combining last two inequality will give ($p < 2+\sqrt{\frac{2}{n-1}}$ needed here)

$$\int_\Sigma |A|^{2p}\phi^2 \le C(p)\int_\Sigma |A|^{2p-2}|\nabla\phi|^2$$

- Change $\phi \to \phi^p$ and use Hölder inequality to get the final result.

$$\int_\Sigma |A|^{2p}\phi^{2p} \le C(p)\int_\Sigma |A|^{2p-2}\phi^{2p-2}|\nabla\phi|^2 \le C(p)\left(\int_\Sigma |A|^{2p}\phi^{2p}\right)^{\frac{p-1}{p}}\left(\int_\Sigma |\nabla\phi|^{2p}\right)^{\frac{1}{p}}.$$

From L^p estimates for curvature to Bernstein type theorem

Theorem (Schoen-Simon-Yau)

A complete two-sided stable minimal hypersurface $\Sigma^{n-1} \in \mathbb{R}^n$ with volume growth at most CR^{n-1} and $n \leq 6$ is a hyperplane.

Proof.

Choose ϕ as a cutoff function that supported in B_{2R} and equal to 1 in B_R with gradient bounded by $\frac{C}{R}$, then

$$\int_{B_R} |A_\Sigma|^{2p} \leq \frac{C}{R^{2p}} \mathrm{Area}(\Sigma \cap B_{2R}) \leq CR^{n-1-2p} \to 0$$

when choose $2p > n-1$ as $R \to \infty$. $\square$