Curvature Estimates for Stable Minimal Triple Junction Surfaces

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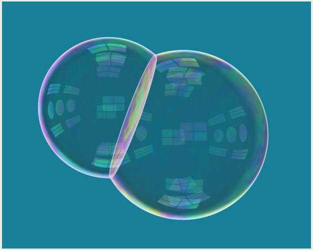
Second Part

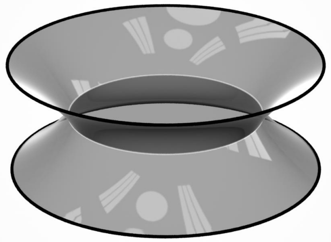
1. Basic Facts about minimal triple junction surfaces
2. Curvature estimate of minimal triple junction surfaces
3. Generalized Bernstein Theorem

Definition (Triple Junction Surfaces)

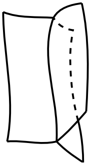
Let $(\Sigma^{(i)}, \partial\Sigma^{(i)})_{i=1,2,3}$ be three surfaces with boundary in $\mathbb{R}^3$. We call the union $\Sigma = \cup_{i=1}^{3} \Sigma^{(i)}$ is a *triple junction surface* in $\mathbb{R}^3$ if these surfaces $\Sigma^{(i)}$ have the same boundary, i.e. $\partial\Sigma^{(1)} = \partial\Sigma^{(2)} = \partial\Sigma^{(3)}$ in $\mathbb{R}^3$. We write Γ as their common boundary $\Gamma = \Sigma^{(i)}$ for $i = 1,2,3$. We write it as (Σ, Γ).

Examples (https://faculty.math.illinois.edu/~jms/Images/)





- **L. Simon (1993):** $C^{1,\alpha}$ regularity near cylindrical tangent cones.
- **B. Krummel (2014):** Higher regularity of minimal submanifolds with common boundary.



Stationary varifold
↓
$C^{1,\gamma}$ regularity
↓
Smooth (Analytic)

- **A. Freier, D. Depner, H. Garcke:** Mean curvature flow with triple junctions.
- **F. Schulze, B. White (2020):** Regularity of mean curvature flow with triple edges.
- A lot of works on the network flow.

Minimal triple junction surfaces in $\mathbb{R}^3$

Definition (Minimal triple junction surfaces)

(Σ, Γ) is called a *minimal triple junction surface* if the mean curvature vector of each piece of surface $\Sigma^{(i)}$ vanishes identically and they make a same angle $\frac{2\pi}{3}$ with each other along Γ. I.e. $\vec{H}_{\Sigma^{(i)}} \equiv 0$ and $\angle(\tau_i, \tau_j) = \frac{2\pi}{3}$ where τ_i being the outer conormal of Γ in $\Sigma^{(i)}$.

Proposition

(Σ, Γ) is a minimal triple junction surface if and only if one of the following holds.

- Σ is a critical point of area function.
- The coordinate the function $x_j^{(i)}$ $j = 1,2,3$ is harmonic on $\Sigma^{(i)}$ and their sum of outer normal derivatives is zero along Γ. I.e. $\Delta_{\Sigma^{(i)}} x_j^{(i)} = 0$ and $\sum_{i=1}^{3} \frac{\partial}{\partial \tau_i} x_j^{(i)} = 0$ for $j = 1,2,3$ along Γ.

First variation formula

$$\delta\Sigma(X) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Area}(\Sigma_t) = -\sum_{i=1}^{3} \int_{\Sigma^{(i)}} \vec{H}_{\Sigma^{(i)}} \cdot X \, dA + \sum_{i=1}^{3} \int_{\Gamma} X \cdot \tau^{(i)} \, ds$$

Some examples of minimal triple junction surfaces

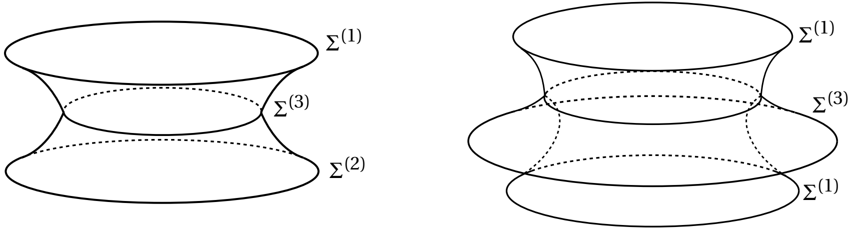


Figure: Two kinds of Y-shaped catenoid

Some examples of minimal triple junction surfaces

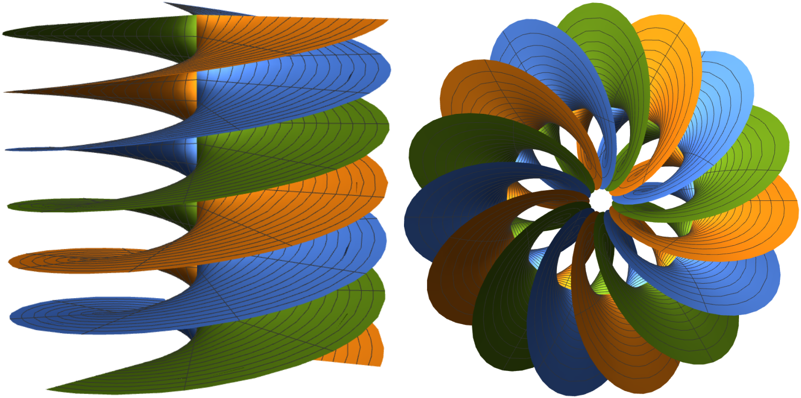


Figure: Y-shaped bent helicoid and usual helicoid

Basic properties of minimal triple junction surfaces

Distance function on Σ

Let $d^{(i)}(\cdot,\cdot)$ be the distance function on $\Sigma^{(i)}$.
Define distance function $d(x,y)$ on Σ by the following

$$d(x,y) := \inf\left\{ \sum_{k=1}^{l-1} d^{(i_k)}(x_k, x_{k+1}) : x_0 = x, x_{l+1} = y, x_1, \cdots, x_l \in \Gamma, \right.$$
$$\left. i_0 = i, i_{l-1} = j, 1 \le i_1, \cdots, i_{l-2} \le 3, \text{ for } l \in \mathbb{N} \right\}$$

Define $d_\Gamma(x) := \inf_{y\in\Gamma} d(x,y)$ for $x \in \Sigma$, $T_r(\Gamma) := \{p \in \Sigma : d_\Gamma(p) < r\}$.

Proposition

There is no compact complete (in the sense of distance d) minimal triple junction surface in $\mathbb{R}^3$.

Function spaces

$$C^k(\Sigma) := \{f = (f^{(1)}, f^{(2)}, f^{(3)}) \in C^k(\Sigma^{(1)}) \times C^k(\Sigma^{(2)}) \times C^k(\Sigma^{(3)})\}.$$
$$C_1^k(\Sigma) := \{f = (f^{(1)}, f^{(2)}, f^{(3)}) \in C^k(\Sigma) : f^{(i)} = f^{(j)}, \forall 1 \le i, j \le 3 \text{ on } \Gamma\}.$$
$$C_2^k(\Sigma) := \{f = (f^{(1)}, f^{(2)}, f^{(3)}) \in C^k(\Sigma) : \sum_{i=1}^3 f^{(i)} = 0 \text{ on } \Gamma\}.$$
$$W^{k,p}(\Sigma) := \{f = (f^{(1)}, f^{(2)}, f^{(3)}) \in W^{k,p}(\Sigma^{(1)}) \times W^{k,p}(\Sigma^{(2)}) \times W^{k,p}(\Sigma^{(3)})\}.$$
$$W_1^{k,p}(\Sigma) := \{f = (f^{(1)}, f^{(2)}, f^{(3)}) \in W^{k,p}(\Sigma) : f^{(i)} = f^{(j)}, \forall 1 \le i, j \le 3 \text{ on } \Gamma\}.$$
$$W_2^{k,p}(\Sigma) := \{f = (f^{(1)}, f^{(2)}, f^{(3)}) \in W^{k,p}(\Sigma), \sum_{i=1}^3 f^{(i)} = 0 \text{ on } \Gamma\}.$$

Examples

- If f is a C^k function in $\mathbb{R}^3$, then $f|_\Sigma := (f|_{\Sigma^{(1)}}, f|_{\Sigma^{(2)}}, f|_{\Sigma^3}) \in C_1^k(\Sigma)$.
- If V is a C^k vector field in $\mathbb{R}^3$, then $V \cdot \nu := (V \cdot \nu^{(1)}, V \cdot \nu^{(2)}, V \cdot \nu^{(3)}) \in C_2^k(\Sigma)$ if $\sum_{i=1}^3 \nu^{(i)} = 0$ along Γ.

Notations and assumption

- From now on, for the triple junction surface (Σ, Γ), we will only consider each $\Sigma^{(i)}$ is orientable and has only one boundary component $\partial\Sigma^{(i)} = \Gamma$.
- We will choose $\nu^{(i)}$ be the normal vector field on $\Sigma^{(i)}$ such that $\sum_{i=1}^{3} \nu^{(i)} = 0$ along Γ. Write $\nu = (\nu^{(1)}, \nu^{(2)}, \nu^{(3)})$ for short.

Integral convention

- For any $f \in C^k(\Sigma)$, we use the following short notations.

$$\int_\Sigma f := \sum_{i=1}^{3} \int_{\Sigma^{(i)}} f^{(i)} dA^{(i)} \qquad \int_\Gamma f := \int_\Gamma \sum_{i=1}^{3} f^{(i)} ds_\Gamma$$

Example.

$$\delta\Sigma(X) = -\int_\Sigma H_\Sigma \phi + \int_\Gamma X \cdot \tau$$

Second variation formula for minimal triple junction surfaces

Let (Σ, Γ) be a minimal triple junction surface in $\mathbb{R}^3$, then

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{Area}(\Sigma_t) = \int_\Sigma |\nabla_\Sigma \phi|^2 - |A_\Sigma|^2 \phi^2 - \boxed{\int_\Gamma \phi^2 \boldsymbol{H}_\Gamma \cdot \tau}$$

with $\phi = X \cdot \nu$ which has compact support, $\boldsymbol{H}_\Gamma$, the geodesic curvature vector of Γ in $\mathbb{R}^3$.

Stable minimal triple junction surfaces

We say $(\Sigma, \Gamma) \subset \mathbb{R}^3$ is a stable minimal triple junction surface if $\frac{\mathrm{d}^2}{\mathrm{d}t^2}|_{t=0} \mathrm{Area}(\Sigma_t) \geq 0$ for any variation Σ_t of Σ with compact support.

Stability inequality

If (Σ, Γ) is a stable minimal triple junction surface in $\mathbb{R}^3$, then for any $\phi \in W_2^{1,2}(\Sigma)$ with compact support,

$$\int_\Sigma |A_\Sigma|^2 \phi^2 + \int_\Gamma \phi^2 \boldsymbol{H}_\Gamma \cdot \tau \leq \int_\Sigma |\nabla_\Sigma \phi|^2$$

Theorem

Suppose (Σ,Γ) *is a stable minimal triple junction surface. Then for any* $\phi \in W^{1,2}(\Sigma)\cap L^\infty(\Sigma)$ *such that* $\operatorname{sign}(\phi)|A_\Sigma|^{p-1}|\phi|^p \in W^{1,2}_2(\Sigma)$, $p\in(1,\frac{5}{4})$, *we have(*C_1, C_2 *does not depend on* p*.)*

$$\int_\Sigma |A_\Sigma|^{2p}|\phi|^{2p} \le C_1\int_\Sigma |A_\Sigma|^{2p-2}|\phi|^{2p-2}|\nabla_\Sigma\phi|^2 + \int_\Gamma\left[\frac{p-1}{2}\left|\frac{\partial}{\partial\tau}\log|A_\Sigma|\right| - \boldsymbol{H}_\Gamma\cdot\tau\right]|A_\Sigma|^{2p-2}|\phi|^{2p}$$

If in addition, $\phi\in W^{1,2p}(\Sigma)\cap L^\infty(\Sigma)$, *then*

$$\int_\Sigma |A_\Sigma|^{2p}|\phi|^{2p} \le C_1\int_\Sigma |\nabla_\Sigma\phi|^{2p} + C_2\int_\Gamma\left[(p-1)\left|\frac{\partial}{\partial\tau}\log|A_\Sigma|\right| - \boldsymbol{H}_\Gamma\cdot\tau\right]|A_\Sigma|^{2p-2}|\phi|^{2p}$$

Sketch proof of L^p estimate

- Change $\phi \to \phi |A|^{p-1}$ in stability inequality $(\int_\Sigma |A|^2 \phi^2 + \int_\Gamma \phi \boldsymbol{H}_\Gamma \cdot \tau \leq \int_\Sigma |\nabla_\Sigma \phi|^2)$ to get

$$\int_\Sigma |A|^{2p} \phi^2 \leq (p-1)^2 \int_\Sigma |A|^{2p-4} |\nabla_\Sigma |A||^2 \phi^2 + |A|^{2p-2} |\nabla_\Sigma \phi|^2 + 2(p-1) \int_\Sigma |A|^{2p-3} \phi \nabla_\Sigma |A| \cdot \nabla_\Sigma \phi - \int_\Gamma |A|^{2p-2} \phi^2 \boldsymbol{H}_\Gamma \cdot \tau.$$

- Multiply Simon's identity $(|\nabla_\Sigma |A||^2 = |A| \Delta |A| + |A|^4)$ by $|A|^{2p-4} \phi^2$ and integrate

$$\int_\Sigma |A|^{2p-4} |\nabla_\Sigma |A||^2 \phi^2 = \int_\Sigma |A|^{2p} \phi^2 - 2\phi |A|^{2p-3} \nabla_\Sigma \phi \cdot \nabla_\Sigma |A| - (2p-3) \int_\Sigma |A|^{2p-4} |\nabla_\Sigma |A||^2 \phi^2 + \int_\Gamma |A|^{2p-2} \frac{\partial}{\partial \tau} \log |A| \phi^2.$$

- Combining last two inequalities will give

$$\int_\Sigma |A|^{2p} \phi^2 \leq 3 \int_\Sigma |A|^{2p-2} |\nabla_\Sigma \phi|^2 + \frac{p-1}{2} \int_\Gamma |A|^{2p-2} \phi^2 \frac{\partial}{\partial \tau} \log |A| - \int_\Gamma |A|^{2p-2} \phi^2 \boldsymbol{H}_\Gamma \cdot \tau.$$

Sketch proof of L^p estimate (continuous)

- Change $\phi \to \mathrm{sign}(\phi)\,|\phi|^p$ in the last inequality (condition for ϕ is $\mathrm{sign}(\phi)\,|A|^{p-1}\,|\phi|^p \in W_2^{1,2}(\Sigma)$)

$$\int_\Sigma |A|^{2p}\,|\phi|^{2p} \le 6\int_\Sigma |A|^{2p-2}\,|\phi|^{2p-2}\,|\nabla_\Sigma \phi|^2 + \int_\Gamma \left[\frac{p-1}{2}\left|\frac{\partial}{\partial\tau}\log|A|\right| - \boldsymbol{H}_\Gamma\cdot\tau\right]|A|^{2p-2}\,|\phi|^{2p}.$$

- If $\phi \in W^{1,2p}(\Sigma)$, we can apply Young's inequality when p small,

$$\int_\Sigma |A|^{2p}\,|\phi|^{2p} \le C_1\int_\Sigma |\nabla_\Sigma \phi|^{2p} + C_2\int_\Gamma \left[\frac{p-1}{2}\left|\frac{\partial}{\partial\tau}\log|A|\right| - \boldsymbol{H}_\Gamma\cdot\tau\right]|A|^{2p-2}\,|\phi|^{2p}.$$

Main Theorem. Generalized Bernstein Theorem

Theorem (Γ compact)

Let (Σ,Γ) be a minimal triple junction surface in $\mathbb{R}^3$. Suppose Σ is complete, stable and has quadratic area growth. Furthermore, we assume Γ is compact, then each $\Sigma^{(i)}$ is flat.

Theorem (Γ straight line)

Let (Σ,Γ) be a minimal triple junction surface in $\mathbb{R}^3$. Suppose Σ is complete, stable and has quadratic area growth. Furthermore, we assume Γ is a straight line, then each $\Sigma^{(i)}$ is flat.

Proof of Theorem when Γ compact

The case of none of $\Sigma^{(i)}$ is flat.

- Write L^p estimate in the following ways,

$$\int_\Sigma |A_\Sigma|^{2p} |\phi|^{2p} \leq C_1 \mathrm{I} + \mathrm{II} - \mathrm{III}$$

$$\mathrm{I} := \int_\Sigma |A_\Sigma|^{2p-2} |\phi|^{2p-2} |\nabla_\Sigma \phi|^2$$

$$\mathrm{II} := \int_\Gamma \frac{p-1}{2} \left| \frac{\partial}{\partial \tau} \log |A_\Sigma| \right| |A|^{2p-2} |\phi|^{2p}$$

$$\mathrm{III} := \int_\Gamma \boldsymbol{H}_\Gamma \cdot \tau \, |A_\Sigma|^{2p-2} |\phi|^{2p}$$

- Fix three nonzero constants $c^{(i)}$ such that $\sum_{i=1}^3 c^{(i)} = 0$. ρ_r, cut-off function supported in $T_{2r}(\Gamma)$ and equal to 1 in $T_r(\Gamma)$. $g^{(i)} = \prod_{j \neq i} \left| A_{\Sigma^{(j)}} \right|$. Choose ϕ as

$$\phi^{(i)} = \mathrm{sign}(c^{(i)}) \left| c^{(i)} \right|^{\frac{1}{p}} \left(\rho_1 (g^{(i)})^{\frac{p-1}{p}} + \rho_r - \rho_1 \right)$$

Check $\mathrm{sign}(\phi) \left| \phi \right|^p |A_\Sigma|^{p-1} \in W_2^{1,2}(\Sigma)$.

Proof of Theorem when Γ compact

Choose of ϕ, $\phi^{(i)} = \operatorname{sign}(c^{(i)}) \left|c^{(i)}\right|^{\frac{1}{p}} \left(\rho_1 (g^{(i)})^{\frac{p-1}{p}} + \rho_r - \rho_1\right)$

- Estimate

$$\mathrm{III} = \int_\Gamma \boldsymbol{H}_\Gamma \cdot \tau |A_\Sigma|^{2p-2} |\phi|^{2p} = \int_\Gamma \boldsymbol{H}_\Gamma \cdot \tau c^2 \prod_{i=1}^{3} |A_{\Sigma^{(i)}}|^{2p-2}$$

  - Adjust $c^{(i)}$ to make $\mathrm{III} \geq 0$.

- Estimate

$$\mathrm{II} = \int_\Gamma \frac{p-1}{2} \left|\frac{\partial}{\partial \tau} \log |A_\Sigma|\right| |A|^{2p-2} |\phi|^{2p}$$

  - Choose p small to make sure $\mathrm{II} < \varepsilon$.

- Estimate

$$\mathrm{I} = \int_\Sigma |A_\Sigma|^{2p-2} |\phi|^{2p-2} |\nabla_\Sigma \phi|^2 = \left(\int_S + \int_{T_2(\Gamma)\setminus S} + \int_{T_{2r}(\Gamma)\setminus T_r(\Gamma)}\right) |A_\Sigma|^{2p-2} |\phi|^{2p-2} |\nabla_\Sigma \phi|^2$$

Split $\mathrm{I} = \mathrm{I}'$(Singularity) $+\mathrm{I}_1$(Regular and near Γ) $+\mathrm{I}_2$(Regular and far from Γ)

  - choose singular region small enough to make sure $\mathrm{I}' < \varepsilon$.
  - choose p closed to 1 enough to make sure $\mathrm{I}_1 < \varepsilon$. From now on, we will fix the choice of p.
  - Choose r large enough to make sure $\mathrm{I}_2 < \varepsilon$.

Special case in the proof.

When one of the $\Sigma^{(i)}$ is flat, says $\Sigma^{(3)}$ is flat, we need to choose $\phi^{(3)} \equiv 0$ to make ϕ satisfies compatible condition. This time we cannot make $\mathrm{III} > 0$ in general since $c^{(i)}$ is basically fixed upto a scaling.
We need the following lemma to deal with this case. (Follow from B. White's estimate of total curvature of surfaces)

Lemma

For each $\Sigma^{(i)}$ with boundary Γ, we have

$$\int_\Gamma -\boldsymbol{H}_\Gamma \cdot \tau^{(i)} \le \int_{\Sigma^{(i)}} -K_{\Sigma^{(i)}}$$

Choice of ϕ, $\phi^{(i)} = \operatorname{sign}(c^{(i)}) \left|c^{(i)}\right|^{\frac{1}{p}} \left(\rho_1 \left|A_{\Sigma^{(j)}}\right|^{\frac{p-1}{p}} + \rho_r - \rho_1\right)$ with $(i,j) = (1,2),(2,1)$

Note

$$\int_\Sigma c^2 |A_\Sigma|^{2p} \simeq \int_\Sigma c^2 |A_\Sigma|^2 = -2\int_\Sigma c^2 K_\Sigma \ge -\int_\Gamma \boldsymbol{H}_\Gamma \cdot \tau c^2 \simeq -\int_\Gamma \boldsymbol{H}_\Gamma \cdot \tau |A_\Sigma|^{2p-2} |\phi|^{2p}$$

We can control III by $\int_\Sigma |A_\Sigma|^{2p} |\phi|^{2p}$ when p small.

Relation with the capillary minimal surface

Corollary

Let P be a plane in $\mathbb{R}^3$. Then there is no (oriented) stable complete minimal surface Σ with boundary $\partial\Sigma \subset P$, $\partial\Sigma$ compact and Σ has quadratic area growth and has angle $\frac{\pi}{3}$ with P along $\partial\Sigma$.

Remark

Recently, I've learned that Han Hong and Artur B. Saturnino gave more precise curvature estimates over capillary surface including capillary minimal surface. Their results are stronger than above corollary.

Thank You!