General Theory of Partial Differential Equations on Triple Junction Surfaces

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Abstract

We develop the fundamental tools from functional analysis and partial differential equations to study the geometric and analytic aspects of triple junction hypersurfaces, a special class of singular manifolds whose boundaries are identified in a particular manner. We define some useful spaces on such singular objects and describe a kind of second-order elliptic operator defined on these function spaces. We extend the standard results in PDE theory for second-order elliptic operators on smooth Riemannian manifolds, including existence, regularity, spectrum theory, etc., to our singular setting. After that, we mention some applications of this theory, including the study of the Morse index for minimal hypersurfaces with triple junctions and the conformal structure on surfaces with triple junctions.

Our new PDE theory is essential to the study of immersed minimal hypersurfaces with triple junctions. In [Wan22], we have observed the appearance of such function spaces as an example. This motivates our study of these function spaces in a more general setting. In [Wan21a, Wan21b], we note that it is vital to have a regularity result so that we can use the powerful tools from elliptic PDEs. This is another motivation for the general theory of elliptic partial differential theory on triple junction hypersurfaces.

Once we have established the regularity, almost all PDE tools can be applied to triple junction hypersurfaces. In particular, we expect these results can also be extended to other geometric settings. For instance, we may consider defining heat-type equations on hypersurfaces with triple junctions. We may also consider more irregular hypersurfaces like surface clusters.

摘要

從泛函分析與偏微分方程中，我們建立了一套基本的工具去從幾何與分析的方面去研究三連接超曲面（triple junction hypersurfaces）的性質。其中三連接超曲面是指一種特殊的奇異流形，它們的邊界以一種特定的方式等同起來。 我們可以在這個奇異流形上面定義一些函數空間以及在這些函數空間一種二階橢圓算子。我們可以把經典二階橢圓方程的結果從光滑流形拓展到奇異流形上，包括解的存在性，唯一性，正則性以及橢圓算子的譜理論等等。 在此之後，我們就可以把它應用到一些地方，比如說我們可以研究極小三連接超曲面的Morse指數與三連接曲面上的共形結構。

我們這些新的偏微分方程理論對於研究浸入的極小三連接超曲面是非常重要的。在[Wan22]中，我們已經觀察到在三連接超曲面上有這些函數空間。這個就是我們要在更一般意義下研究總結這些函數空間的一個動機。另外，在[Wan21a, Wan21b]中，我們也注意到我們也極需要有一套關於橢圓方程解的正則性的結果。這也是我們要建立一個在三連接超曲面上關於橢圓方程理論的一個動機。

一旦我們有了正則化的結果，那麼幾乎所有偏微分方程的工具可以應用到三連接超曲面上了。特別的，我們也希望這些結果可以進一步拓展到一些幾何情形。比如說我們可以考慮在三連接超曲面上面定義熱方程類型的方程。我們也可以考慮其他一些正則性更差的超曲面。

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Acknowledgments

I would like to thank my supervisor Professor Martin Man Chun Li. He led me into the world of minimal surfaces and geometric measure theory. During the past few years, I have learned geometric measure theory under the supervision of Professor Li. Under the guidance of Professor Li, I completed three papers during my Ph.D. With his continuous support and encouragement, I can successfully complete my papers smoothly. Besides, much of the content of this thesis is also based on previous discussions with Professor Li. Discussing with him deepened my understanding of minimal surfaces while letting me know how to conduct mathematical research on my own.

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Chapter 1

Introduction

Minimal surfaces are important objects in mathematics due to their natural phenomenon arising from certain area minimizing property. In this thesis, I will focus on the singular version of minimal surfaces, namely the minimal triple junction surfaces.

The research on minimal surfaces went back to J.L. Lagrange, who wanted to find a surface with the least area with a prescribed boundary. In general, we may obtain a singular minimal surface with a given boundary after minimizing. This phenomenon was observed by J. Plateau, who experimented with soap films, and he formulated the well-known Plateau's laws to describe their structures.

Since the triple junction structure naturally arises from the Plateau's laws, there are quite a lot of results related to triple junction surfaces. J. Taylor [Tay73, Tay76] has studied specific minimizing problems and found two types of singularities would appear. One is the Y-type singularity related to the triple junction surfaces. Furthermore, the work of G. Lawlor and F. Morgan [LM96] has shown that triple junction surfaces are always locally area minimizing in arbitrary dimension and codimension. From the point of view of mappings (i.e., [Dou31]), C. Mese and S. Yamada [MY06] tried to minimize the Dirichlet energy to get the Y-type singularities. Recently, J. Bernstein and F. Maggi [BM21] showed the rigidity of the Y-shaped catenoid, a natural extension of the result in [Sch83].

The regularity of stationary integral varifolds is still a central problem in geometric measure theory. L. Simon [Sim93] showed if a stationary integral varifold is very close to the union of three half-planes, which is stationary in $\mathbb{R}^n$, then locally, this varifold is indeed a $C^{1,\mu}$ triple junction minimal submanifold. B. Krummel [Kru13] improved the regularity from $C^{1,\mu}$ to C^∞ near their junctions.

Just like the classical mean curvature flows, we can also consider moving the triple junction surfaces by their mean curvatures. Recently, there have been some interesting progress in this area, e.g., [Fre10, DG13, DGK14, SW20].

Hence, the topics about triple junction surfaces are pretty interesting and attractive to many researchers. In this thesis, I will introduce the notions of intrinsic triple junction hypersurfaces. We will focus on the triple junction hypersurfaces that can arise from the embedded triple junction hypersurfaces in an ambient Riemannian manifold.

1.1 Summary of main results

In this section, I will briefly summarize the main results and explain why we want to study the intrinsic triple junction hypersurfaces and the elliptic operators on them.

1.1.1 Curvature estimate and generalized Bernstein theorem for stable triple junction surfaces

The first thing I have investigated is the curvature estimate for stable triple junction surfaces. Usually, the curvature estimate plays a key role in the theory of minimal surfaces, especially in the min-max theory (e.g., [Smi83, CdL03, DLT13, LR18, LZ21, LZZ21]).

Besides, the curvature estimate for stable hypersurfaces itself is an interesting topic. Note that curvature estimate for stable minimal hypersurfaces is equivalent to the generalized Bernstein theorem for a stable minimal hypersurface in $\mathbb{R}^{n+1}$.

The classical and powerful results are due to Schoen, Simon, Yau [SSY75]. Their methods are also used in another type of curvature estimate. For instance, the work [LZZ21] used this trick to show that a particular curvature estimate holds for minimal capillary hypersurfaces.

After that, Schoen and Simon [SS81] developed a deep theory about the regularity and compactness theory for stable minimal hypersurfaces. This will provide us with an optimal dimension estimate of the singular set, the validity of curvature estimate, and the generalized Bernstein theorem hold. Note that all of the previous results require volume growth conditions.

If we only focus on the surface case, there are other ways to prove the curvature estimate and generalized Bernstein theorem without the area growth condition. Interested readers may refer to the work of [dCP12, Pog81, FCS80].

In particular, the recent work of Chodosh and Li [CL21] shows we can get the curvature estimate for stable minimal hypersurfaces in $\mathbb{R}^4$ without the volume growth condition.

In [Wan22], we can use the trick due to Schoen, Simon, and Yau to show that if a triple junction surface Σ is stable with quadratic area growth condition, and the triple junction is compact, then we can show Σ is indeed flat. Since there are a lot of notations we need to define, we put the precise statement in Subsection 6.1.2.

Note that there are several particularly interesting function spaces during the study of stability inequality in this work. For example, suppose $\Sigma = (\Sigma^1, \Sigma^2, \Sigma^3)$ is a triple junction hypersurface. The function $f = (f^1, f^2, f^3)$ in this space will have sum of f^i restricted on boundary vanishes. Hence, it is worth giving more precise definitions of these function spaces and studying the elliptic operators defined on these spaces. Note that we also need to develop the regularity theory to give more precise definitions and properties of the Morse index of minimal triple junction hypersurfaces.

This is one motivation why we need to develop a theory about elliptic partial differential equations on triple junction hypersurfaces.

1.1.2 Computing the index of stationary networks in $\mathbb{S}^2$

Before moving to the Morse index of triple junction hypersurfaces, we can first investigate the Morse index of networks. This is because the regularity becomes trivial for networks.

By the way, the models of geodesic and stationary networks can be viewed as the simpler version of minimal surfaces and stationary varifolds. There are still many active researches focusing on this direction. In particular, the first step towards solving the double bubble conjecture is the planar double bubble conjecture, which was solved by a group of undergraduate students [FGB+93], and later on, the stable case was solved by [MW02]. There are also many researches about planar clusters (e.g., [CHH+94, MFG98, HM05, Wic04]). In particular, in the work of [MY06], the authors also considered the 1-dimensional case first.

In [Wan21a], we will define the stability operators based on the second variation formula and define the Morse index of these networks. Since these definitions require much space, we will illustrate only one of the main results in this paper as follows.

Theorem 1.1. *The Morse index of all closed stationary triple junction networks in $\mathbb{S}^2$ is $F-1$ where the F is the number of regions on $\mathbb{S}^2$ cut out by this network. The corresponding eigenfunctions are all locally constant functions.*

Since the stability operator is different from the Laplacian on networks in a sphere only up to a constant, we may confirm one of Yau's open problems [Yau82] in the case of triple junction networks. Precisely, our result shows that the first non-trivial eigenvalue for Laplacian on triple junction networks on $\mathbb{S}^2$ is 1.

The method we used in the proof of Theorem 1.1 is to divide a big network into several smaller networks, and we construct a Dirichlet-to-Neumann map with respect to this subdivision. Then we can compute the Morse index of the whole network using the Morse index and nullity of smaller networks and the index of the Dirichlet-to-Neumann map. This will significantly reduce the computational time since the original networks might be very complicated.

Moreover, this method can also be easily generalized to higher dimensions and help us to compute the Morse index of triple junction hypersurfaces. We will go back to the Dirichlet-to-Neumann map on triple junction hypersurfaces in Section 6.2 after we have developed the regularity theory.

1.1.3 Conformal structures on triple junction surfaces

In the classical theory of minimal surfaces, the conformal structures on minimal surfaces are pretty helpful. In particular, observing the Gauss map of a minimal surface immersed in $\mathbb{R}^3$ is a conformal map, we can determine the conformal structures for a complete minimal surface in $\mathbb{R}^3$ with finite total curvature [Oss86]. There is a survey [MP04] that investigates the conformal properties of minimal surfaces.

Hence, it is worth studying the conformal structure on triple junction surfaces. We will focus on this topic in Chapter 7. In general, we can only develop such a theory after solving the regularity problem. However, the weak uniformization does not require the regularity result in one particular case. In [Wan21b], we develop a weak uniformization for some specific triple junction surfaces. As it needs some notations to precisely state this theorem, I will put the details in Section 7.4.

The tools used in the paper [Wan21b] is the uniformization of surfaces with boundaries. Indeed, we extend Rupflin's work [Rup21] to ensure it can be applied to triple junction surfaces in this particular case.

Besides, the uniformization of surfaces with boundaries itself is an interesting problem. There are a lot of related works like [OPS88, OPS89, Khu91, Kim08]. Moreover, Brendle [Bre02a, Bre02b] has considered a family of curvature flow of metrics and found that the limit metrics have constant Gaussian curvature and constant geodesic curvature on the boundary. This provides us with another way to think about uniformization.

1.2 Outline of this thesis

In this section, we will describe the organization of this thesis.

In Chapter 2, we will fix notations and introduce the concepts of triple junction hypersurfaces using triple junction structures. We will also give some standard examples to illustrate the triple junction hypersurfaces.

In Chapter 3, we define the function spaces, vector fields, differential forms, and metrics on triple junction hypersurfaces. These definitions are all required to be compatible with the triple junction structure. In particular, these spaces share some common properties with the usual spaces that appear on smooth hypersurfaces. We will use several examples to illustrate what these spaces will look like and demonstrate they are the spaces we want if the triple junction hypersurfaces are embedded in a larger smooth Riemannian manifold.

In Chapter 4, we define the Sobolev-type spaces and second-order elliptic operators on triple junction hypersurfaces. In particular, the stability operator used in [Wan22] will automatically be one of such operators.

From now, we have all the concepts to define elliptic partial differential equations on triple junction hypersurfaces. This is done in Chapter 5. In this chapter, we will define the weak solutions to the elliptic equations and establish the regularity of such solutions. In particular, we will mainly focus on the regularity near triple junctions. After establishing the regularity results, it is standard to develop the spectrum theory for second-order elliptic operators.

In Chapter 6, we will describe a method of computing the index and nullity for triple junction hypersurfaces. As we have mentioned before, this is an extension of my previous work [Wan21a]. We will find that the index for the second-order elliptic operators can also be computed through the information on each hypersurface and Dirichlet-to-Neumann map. At last, we will give an example of its application.

The last chapter will study the conformal structure on triple junction surfaces. We will show that we can always find a metric with constant Gaussian curvature on triple junction surfaces when $\chi(\Sigma) \leq 0$. We also introduce weak uniformization and use an example to illustrate why we consider weak uniformization. At last, we mention the relation between extremal metrics for first non-trivial eigenvalues and conformal structures.

Chapter 2

Basic notations and definitions

2.1 Notations

Let us fix an integer $n \geq 2$ as the dimension of hypersurfaces. We adapt the following notations.

- $\mathbb{R}^n_+ := \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$.
- $\Gamma(E)$, the collection of smooth sections for vector bundle $E \to \Sigma$ or a rough section define a.e. on Σ depending on the context. Usually, if E is a subbundle of $T\Sigma$, we will use $\Gamma(E)$ to denote the set of smooth sections. If we view E as a vector-valued function, we do not expect the section to be smooth all the time. It may only be defined almost everywhere with respect to the measure on Σ. (See Section 4.1 about Sobolev spaces on triple junction hypersurfaces for details.)
- We will use Einstein summation if needed.
- $\langle X, Y \rangle_g$ denotes the inner product of X, Y under metric g. We omit g if the metric is clear from the context. Sometimes, we use the alternative notation $g(X, Y)$ to emphasize that g may be changing. If X, Y are vector fields on $\mathbb{R}^n$ with standard Euclidean metric, we use $X \cdot Y$ to denote their inner product.

- ∇_Σ, ∇^g denote the covariant derivative on Σ with respect to the metric g. Sometimes we will omit Σ or g if it is clear from the context.

- Ric^N denotes the Ricci curvature on N. We will write $\mathrm{Ric}^N(X) := \mathrm{Ric}^N(X, X)$. Here, N is a Riemannian manifold.

- $\mathrm{div}_\Sigma X := \sum_{i=1}^n \langle \nabla_{e_i} X, e_i \rangle$ is defined as the divergence of a vector field, where $\{e_i\}$ is the orthonormal basis of a tangent plane. Note that we can also define the divergence for a tensor T as $(\mathrm{div}_\Sigma T)_I := T_{Ik,k}$ for multi-index I.

- $\Delta_\Sigma f := \mathrm{div}_\Sigma \nabla_\Sigma f$ denotes the Laplacian operators acting on a function f.

Suppose Σ is smoothly immersed into an $(n+1)$-dimensional Riemannian manifold (N, g_N), then we will use the following notations related to submanifolds.

- ν is the unit normal vector field on Σ when we immerse Σ into an $(n+1)$-dimensional Riemannian manifold.

- $A_\Sigma(X, Y) := \left\langle \nabla_X^{g_N} Y, \nu \right\rangle$ denotes the second fundamental form of Σ.

- $H_\Sigma := \sum_{i=1}^n A_\Sigma(e_i, e_i)$ denotes the mean curvature of Σ, where we choose $\{e_i\}$ as the orthonormal basis at a tangent plane. We will write $\boldsymbol{H}_\Sigma := \sum_{i=1}^n \nabla_{e_i}^{g_N} e_i$ as the mean curvature vector of Σ. Note that $\boldsymbol{H}_\Sigma$ can be defined in any codimension.

In the later definitions, we will use a tuple to denote the ordered set of functions, vector fields, differential forms, metrics, etc., on triple junction hypersurfaces. When we talk about the actions between these tuples, we always mean we will separate them and do those actions on each hypersurface and combine their results to get another tuple.

For example, if $f = (f^1, f^2, f^3)$ and $g = (g^1, g^2, g^3)$, we will write $fg := (f^1 g^1, f^2 g^2, f^3 g^3)$. Similarly for any other kinds of actions like if $X = (X^1, X^2, X^3)$ is a tuple of vector fields, then $\frac{\partial}{\partial X} f = \left(\frac{\partial}{\partial X^1} f^1, \frac{\partial}{\partial X^2} f^2, \frac{\partial}{\partial X^3} f^3 \right)$.

2.2 $C^{(0)}$ triple junction hypersurfaces

For each $i = 1, 2, 3$, we write Σ^i as an n-dimensional smooth manifold with smooth boundary $\partial\Sigma^i$. In this thesis, we will assume each Σ^i is always connected and orientable. In general, $\partial\Sigma^i$ may have more than one component. We will write $\partial\Sigma^i = \Gamma^i \sqcup \mathring{\Gamma}^i$ as the disjoint union of the components of $\partial\Sigma^i$. We want to identify their interior boundary $\mathring{\Gamma}^i$ to get a triple junction manifold. We call it triple junction hypersurfaces since we will only focus on the triple junction manifolds that can be immersed into a larger ambient manifold.

We write Γ as an $(n-1)$-dimensional smooth manifold without boundary.

Definition 2.1. We say $\Sigma = (\Sigma^1, \Sigma^2, \Sigma^3)$ is a $C^{(0)}$ *(intrinsic) triple junction hypersurface*, if there exists an $(n-1)$-dimensional manifold Γ and three diffeomorphisms $\varphi^i : \Gamma \to \mathring{\Gamma}^i$, $i = 1, 2, 3$. We will call Γ as the triple junction of Σ and we will write $\partial\Sigma = \bigcup_{i=1}^3 \Gamma^i$.

Under this definition, we can write $\Gamma = \mathring{\Gamma}^i$ for $i = 1, 2, 3$. We call these conditions (i.e., the diffeomorphism φ_i) as the $C^{(0)}$ triple junction structure on Σ. Note that Σ will automatically have the quotient topology. For example, for any $p \in \Gamma$, if we say U is a neighborhood of p, we actually means $U = (U^1, U^2, U^3)$ such that U^i is a neighborhood of $p \in \mathring{\Gamma}^i$ in Σ^i.

Note that we may allow Γ^i or $\mathring{\Gamma}^i$ to be an empty set. If $\mathring{\Gamma}^i = \emptyset$, then Σ will be the union of three independent manifolds. In this thesis, we will only focus on the case $\Gamma \neq \emptyset$, although almost all results will trivially hold for the case $\Gamma = \emptyset$. On the other hand, if $\Gamma^1 = \Gamma^2 = \Gamma^3 = \emptyset$, we will write $\partial\Sigma = \emptyset$.

Remark 2.2. *$\partial\Sigma$ does not play any roles in the definitions of triple junction hypersurfaces. So it is harmless to temporarily assume $\partial\Sigma = \emptyset$. We need it nonempty if we want to solve a PDE problem on Σ. In that case, we need Σ to be compact and hence we also need to consider the case $\partial\Sigma \neq \emptyset$ in order to impose some boundary conditions.*

Remark 2.3. *In general, if some of Σ^i has more than one component, we can define the triple junction hypersurface as above. Sometimes Σ can become connected under the quotient topology. Almost all the results in this thesis are valid for this case. See Subsection 7.3 for an example.*

Definition 2.4. We say Σ is a C^0 extrinsic (non-degenerate) triple junction hypersurface if there exists an $(n+1)$-dimensional manifold N and three smooth immersions $\phi^i : \Sigma^i \to N$ such that the following holds,

- ϕ^i restricts on $\mathring{\Gamma}^i$ is a diffeomorphism to an immersed $(n-1)$-dimensional submanifold Γ of N for different i.
- For each $p \in \Gamma$, $T_p\Sigma^i \neq T_p\Sigma^j$ for every $1 \leq i \neq j \leq 3$.

It is easy to note if Σ is a C^0 extrinsic triple junction hypersurface, then it will automatically be an intrinsic one. The non-degenerate condition (the second condition in Definition 2.4) makes sure that Σ^i intersects Σ^j along Γ transversally for different i, j. In this thesis, we will always assume Σ is intrinsic unless otherwise stated.

Definition 2.5. We say a triple junction hypersurface Σ has density θ, if θ can be written as $\theta = (\theta^1, \theta^2, \theta^3)$ and each θ^i is a positive constant function on Σ^i.

In this thesis, we will always assume Σ has density θ. Indeed, it is applicable only when we immerse it into a larger ambient manifold N and try to compute its area. The density will determine how we can define the function spaces and several other spaces.

2.3 Triple junction structure

Now let us define the triple junction structure on Σ to make it into a triple junction hypersurface.

Before giving the precise definitions, let us recall some concepts of distributions and foliations on manifolds.

Suppose M is an n-dimensional smooth manifold. Let τ be a vector field on M and D be an $(n-1)$-dimensional smooth tangent distribution on M. We suppose $\tau \notin D$. Recall we say D is a distribution if it is a subbundle of tangent bundle $T\Sigma$.

Definition 2.6. We say (τ, D) is *integrable* if D is involutive and for any $V \in \Gamma(D)$, we have $[\tau, V] \in \Gamma(D)$.

Recall that we say D is involutive if for any $V, W \in \Gamma(D)$, we have $[V, W] \in \Gamma(D)$. Standard proof for the Frobenius Theorem implies we can describe the structure of (τ, D) using local coordinate charts.

Proposition 2.7. *Suppose (τ, D) is integrable on M, then for any $p \in M$, we can find a coordinate chart $(x_1, \cdots, x_n)$ near p such that*

- $\tau = \frac{\partial}{\partial x_1}$ *near p,*
- *D is spanned by $\frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}$ near p.*

From Proposition 2.7, we know (τ, D) will determine a foliation $\{x_1 = s\}$ near p and D_q will become the tangent space of some leaves for each q near p.

Now let us go back to the C^0 triple junction hypersurface Σ.

Definition 2.8. Let τ^i be a smooth vector field near Γ on Σ^i and D^i an $(n-1)$-dimensional smooth tangent distribution near Γ on Σ^i. We suppose each τ^i is pointing outside of Γ and $D^i = T\Gamma$ when restricted on Γ. Then we write $(\tau, D) = ((\tau^1, \tau^2, \tau^3), (D^1, D^2, D^3))$ and call it as the *triple junction structure* on Σ if each (τ^i, D^i) is integrable on Σ^i for $i = 1, 2, 3$.

We say Σ is a *triple junction hypersurface* if we assume Σ is a $C^{(0)}$ triple junction hypersurface together with a triple junction structure (τ, D).

Remark 2.9. *Note that we only require τ and D to be defined near Γ. This is because, by Proposition 2.7, the existence of D is equivalent to the existence of the foliation of Γ. Usually, we do not expect the foliation can cover the whole hypersurface.*

Remark 2.10. *(τ, D) can be understood as the $C^{(1)}$ triple structure condition on Σ if we want to compare the previous $C^{(0)}$ triple junction structure. Hence, we say (τ, D) is a triple junction structure on Σ, we already assume we have fixed $C^{(0)}$ triple junction structure on Σ.*

Note that by Proposition 2.7, we can find a good coordinate chart near each $p \in \Gamma$.

Proposition 2.11. *For each $p \in \Gamma$, we can find a local coordinate chart $\varphi : U \to V$ (i.e., $\varphi^i : U^i \to V^i \subset \mathbb{R}^n_+$ for $i = 1, 2, 3$) near p such that*

- *$\varphi(\Gamma \cap U) = \{x_1 = 0\} \cap V$.*

- *$\varphi^i(p) = 0$ and $\varphi^i(q) = \varphi^j(q), \forall q \in \Gamma$ if we identify V^i, V^j as a subset of $\mathbb{R}^n_+$.*

- *$\tau = -\frac{\partial}{\partial x_1}$ near p.*

- *D is spanned by $\frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}$ near p.*

Here, we use the notation $\varphi : U \to V$ to mean $\varphi^i : U^i \to V^i$ where U^i is the neighborhood of p in Σ^i and V^i is the neighborhood of 0 in $\mathbb{R}^n_+$. Similar rules apply for other notations, e.g., $\tau = -\frac{\partial}{\partial x_1}$ actually means $\tau^i = -(\varphi^i)^{-1}_* \left(\frac{\partial}{\partial x_1}\right)$.

Proof of Proposition 2.11. The third and fourth results are clear by Proposition 2.7. For the first one, we note the integral manifold for D^i through p is Γ on Σ^i. On the other hand, the integral manifold for the distribution spanned by $\left\{\frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right\}$ in $\mathbb{R}^n$ is the hyperplane defined by $\{x_1 = c\}$. Hence, we know the coordinate chart defined by Proposition 2.7 will map Γ to the subset of $\{x_1 = c\}$ for $c \in \mathbb{R}$. We can choose $c = 0$ and get $\varphi(\Gamma \cap U) = \{x_1 = 0\} \cap V$. For the second one, we only need to reparametric the coordinate along Γ to ensure the image of Γ under φ^i can be matched. □

Remark 2.12. *Although we use (τ, D) to define the triple junction structure, we will find we can only use τ to define the triple junction structure. This can be seen by the following steps.*

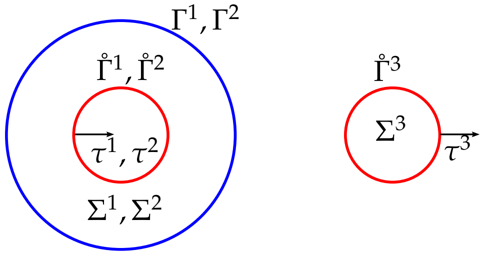
Given $\tau = (\tau^1, \tau^2, \tau^3)$ defined in Definition 2.8, we can consider the smooth flow $\theta^i_t : \Sigma^i \to \Sigma^i$ generated by τ^i (We might extend Σ^i a bit to make it into a manifold without boundary) near Γ. Then we may define $D^i_q := T_q(\theta^i_{t_q\#}\Gamma)$ for q such that there is a t_q with $q \in \theta^i_{t_q\#}\Gamma$. Roughly speaking, we can construct a local foliation of Γ using the flow generated by τ, then the distribution will be taken as the tangent space of this foliation. Note that such D is uniquely determined by τ.

Moreover, we will find we can only choose τ defined on Γ to define the triple junction structure. See Section 3.3 for details.

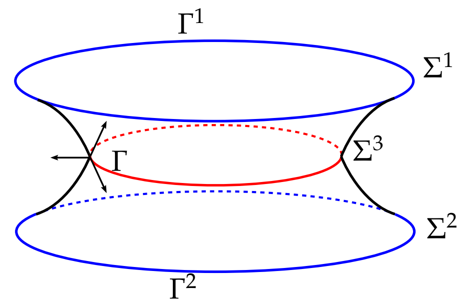
We will keep D in our definitions since we need to use it a lot later on.

2.4 Some examples

Here are some examples of triple junction surfaces (2-dimensional triple junction hypersurfaces).



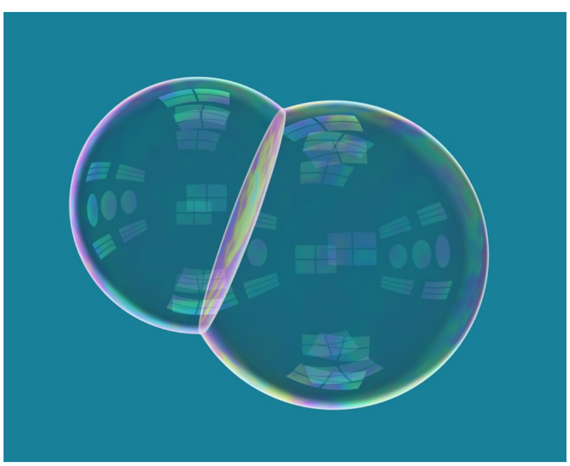
An intrinsic triple junction hypersurface



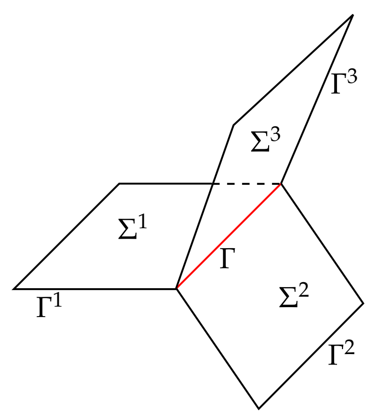
Y-catenoid Embedded in $\mathbb{R}^3$

Figure 2.1: *Y-shaped catenoid*

The first one is the Y-shaped catenoid. As shown in Figure 2.1, the blue curves are the component of $\partial\Sigma$, and the red curves are the interior boundary of Σ^i. We will choose τ^i as the unit outer normal of Γ in Σ^i and let such τ be the triple junction structure on Σ (see Remark 2.12).



(a) *Double Bubble (credit to `https://faculty.math.illinois.edu/~jms/Images/`)*



(b) *Triple Junction Surfaces with Corners*

Figure 2.2: *Some more examples*

Double bubble is another example we might be familiar with since it can be observed in soap bubbles, as shown in Figure 2.2a. In this example, we find each Σ^i is a disk, and the $C^{(0)}$ triple junction structure is just some diffeomorphisms along their boundaries.

There are also many irregular triple junction surfaces. For instance, we can consider the triple junction surface with corners as shown in Figure 2.2b. We suppose each Σ^i is a rectangle region (identical to each other) in $\mathbb{R}^2$. Then we can identify one of their boundaries. In this thesis, I will not consider this case. This is because we need plenty of additional technical definitions and statements to make it well-defined, especially when studying the regularity of solutions.

On the other hand, there are also a lot of 1-dimensional triple junction hypersurfaces, known as networks. In [Wan21a], I have described their structure carefully and given the ideas of how to solve a PDE on the networks.

Chapter 3

Geometric spaces on triple junction hypersurfaces

3.1 Function Spaces on Σ

We fix a triple junction structure (τ, D) on Σ.

Definition 3.1. We say $f = (f^1, f^2, f^3)$ is a *smooth function* on Σ, and write $f \in C^\infty(\Sigma)$, if for each $i = 1, 2, 3$, $f^i : \Sigma^i \to \mathbb{R}$ is a smooth function on Σ^i (smooth up to the boundary).

Roughly speaking, C^∞ function space has no connection with any triple junction structure on Σ. Now let us define some function spaces related to the $C^{(0)}$ structure on Σ. Note that for any $f \in C^\infty(\Sigma)$, we know the restriction of f on Γ can be understood as the smooth map from Γ to $\mathbb{R}^3$. In other words,

$$f|_\Gamma = (f^1|_\Gamma, f^2|_\Gamma, f^3|_\Gamma) \in C^\infty(\Gamma, \mathbb{R}^3).$$

Note that we can also view $C^\infty(\Gamma, \mathbb{R}^3)$ as a set of all the smooth sections of the trivial vector bundle $\Gamma \times \mathbb{R}^3$.

Let $E \subset \Gamma \times \mathbb{R}^3$ be a smooth rank k subbundle of $\Gamma \times \mathbb{R}^3$ for some $k \in \{0, 1, 2, 3\}$. We write $\Gamma(E)$ as the set of all possible sections of E. We do not require the sections in $\Gamma(E)$ to be smooth. In the later definitions about Sobolev spaces, we only require it is defined almost everywhere in $\Gamma(E)$ with respect to the canonical measure on Γ.

We define the $C_E^{(0)}(\Sigma)$ related to E as follows.

Definition 3.2. We say $f \in C_E^{(0)}(\Sigma)$ if $f \in C^\infty(\Sigma)$ and in addition, $f|_\Gamma$ is a smooth section of E.

Note that since $f|_\Gamma$ is smooth for any $f \in C^\infty(\Sigma)$, we can only require $f|_\Gamma \in \Gamma(E)$ instead of requiring $f|_\Gamma$ to be a smooth section in Definition 3.2.

One special case is that we can choose $E = \{(p,(g^1,g^2,g^3)) \in \Gamma \times \mathbb{R}^3 : g^1 = g^2 = g^3\}$. If $f \in C_E^{(0)}(\Sigma)$, then we actually mean f^i will become the same function when restricted on Γ. One can understand $C_E^{(0)}(\Sigma)$ as some kind of continuous function spaces on Σ across the junction.

Let us choose another $E' \subset \Gamma \times \mathbb{R}^3$, a smooth l-dimensional subbundle with $l \in \{0,1,2,3\}$. We define the new subspace of $C^\infty(\Sigma)$ related to E, E' as follows.

Definition 3.3. We say $f \in C_{E,E'}^{(1)}(\Sigma)$ if $f \in C_E^{(0)}(\Sigma)$ and in addition, $\frac{\partial f}{\partial \tau}|_\Gamma$ is a smooth section of E'.

Here, we write $\frac{\partial f}{\partial \tau} = \left(\frac{\partial f^1}{\partial \tau^1}, \frac{\partial f^2}{\partial \tau^2}, \frac{\partial f^3}{\partial \tau^3}\right)$.

Remark 3.4. *Note that $f \in C_{E,E'}^{(1)}(\Sigma)$ means we impose some conditions on the boundary values of f and the first derivative values of f on Γ, so we may understand this space as the C^1 function space across the junction.*

There are two important subbundles we might need to use a lot. We write them as

$$E_1 := \left\{(p,(g^1,g^2,g^3)) \in \Gamma \times \mathbb{R}^3 : g^1 = g^2 = g^3\right\},$$
$$E_\theta := \left\{(p,(g^1,g^2,g^3)) \in \Gamma \times \mathbb{R}^3 : \theta^1 g^1 + \theta^2 g^2 + \theta^3 g^3 = 0\right\},$$

where $\theta = (\theta^1, \theta^2, \theta^3)$ is the density function.

If we put a metric on the bundle $\Gamma \times \mathbb{R}^3$ using density defined as

$$g \cdot g' = \sum_{i=1}^{3} \theta^i g^i (g')^i, \quad \forall g, g' \in (\Gamma \times \mathbb{R}^3)_p \simeq \mathbb{R}^3, p \in \Gamma.$$

Then for any smooth subbundle $E \subset \Gamma \times \mathbb{R}^3$, we can define the orthogonal complement bundle,

$$E^{\perp} := \left\{ (p, g) \in \Gamma \times \mathbb{R}^3 : g \perp E_p, \forall p \in \Gamma \right\}.$$

One particular case is $E_\theta = E_1^{\perp}$.

Now, let us give some examples to help us better understand the above function spaces.

3.1.1 Examples

Let N be an $(n+1)$-dimensional Riemannian manifold. Suppose $\phi : \Sigma \to N$ is a smooth minimal immersion with density $\theta = (\theta^1, \theta^2, \theta^3)$. In other words, we suppose the following conditions hold,

- Each ϕ^i is a minimal immersion from Σ^i to N and $\phi^i|_\Gamma \equiv \phi^j|_\Gamma$ for $1 \le i, j \le 3$.
- Let τ^i be the outer conormal of $\Sigma^i \subset N$ along Γ. Then

$$\sum_{i=1}^{3} \theta^i \tau^i = 0.$$

We will go back to these equivalence conditions when computing the first and second variation formulas for triple junction hypersurfaces (see Section 4.4).

Note that we have a canonical way to choose the triple junction structure (τ, D) using the immersion ϕ^i and a metric on N. We will illustrate this structure later on (Subsection 3.4.3). Now let us just assume we have constructed the triple junction structure (τ, D) such that $\tau^i|_\Gamma$ is the outer unit normal of Γ for $\Sigma^i \subset N$.

The first interesting function space on Σ is the restriction of smooth functions $h \in C^\infty(N, \mathbb{R})$ to Σ. In other words, we choose $f = (f^1, f^2, f^3)$ such that $f^i = h|_{\Sigma^i}$. Clearly, we will find $f \in C^{(0)}_{E_1}(\Sigma)$. Moreover, we note

$$\sum_{i=1}^{3} \theta^i \frac{\partial}{\partial \tau^i} f^i \equiv 0,$$

by the definition of minimal immersion. Then we know $\frac{\partial}{\partial \tau} f|_\Gamma \in \Gamma(E_\theta)$ and therefore, $f \in C^{(1)}_{E_1, E_\theta}(\Sigma)$.

Another interesting function space on Σ is the inner product of a smooth vector field on N with the unit normal vector field along Σ. Let X be a smooth vector field on N. Let $\nu = (\nu^1, \nu^2, \nu^3)$ be the unit normal vector field on Σ, i.e., ν^i is the unit normal vector field on Σ^i, such that $\sum_{i=1}^{3} \theta^i \nu^i = 0$. (In general, we might not able to find a global ν^i to make this equality hold, especially where Γ has more than one component. We will find a more precise description in Subsection 3.1.3.) For such ν on Σ, we can easily find $\langle \nu, X \rangle \in C^{(0)}_{E_\theta}(\Sigma)$ where $\langle \cdot, \cdot \rangle$ is the metric on N and $\langle \nu, X \rangle := (\langle \nu^i, X \rangle)_{i=1}^{3}$. Usually, this function space arises from the variations of Σ. Hence, such space plays a crucial role in studying stability operators. Readers may find similar definitions of these spaces in [Wan22, Wan21a].

3.1.2 Extensions of functions on extrinsic triple junction hypersurfaces

From the above examples, we see one motivation for studying these function spaces is that they come from the restriction of smooth functions in the ambient space.

A natural extension of $C^{(1)}_{E,E'}(\Sigma)$ is to define

$$C^{(k)}_{E_0,E_1,\cdots,E_k}(\Sigma) := \left\{ f \in C^\infty(\Sigma) : \frac{\partial^j f}{\partial \tau^j} \in E_j, \forall 0 \leq j \leq k \right\}.$$

for some smooth subbundles $E_0, \cdots, E_k \subset \Gamma \times \mathbb{R}^3$. The functions in these spaces can be viewed as the C^k differential functions across the junction. However, we find these spaces are not necessary. We will give an example to show for every function $f \in C^{(1)}_{E_1,E_\theta}(\Sigma)$ and Σ is an extrinsic triple junction hypersurfaces, then f is a restriction of a smooth function in the ambient manifold.

Let Σ be three rays $[0, +\infty)$ with their end point $\{0\}$ identified . We can identify the Σ with the set $R_x \cup R_y \cup R_{xy} \subset \mathbb{R}^2$ where

$$\Sigma^1 = \left\{ (x, 0) \in \mathbb{R}^2 : x \geq 0 \right\},$$
$$\Sigma^2 = \left\{ (0, y) \in \mathbb{R}^2 : y \geq 0 \right\},$$
$$\Sigma^3 = \left\{ (x, x) \in \mathbb{R}^2 : x \leq 0 \right\}.$$

Note that $|\Sigma^1| + |\Sigma^2| + \sqrt{2} |\Sigma^3|$ is stationary in the sense of varifold.

Hence if $f \in C^\infty(\mathbb{R}^2)$, we know $f|_\Sigma \in C^{(1)}_{E_1,F}(\Sigma)$ where $F \subset \Gamma \times \mathbb{R}^3$ is defined as

$$F := \left\{ \left(p, (g^1, g^2, g^3)\right) \in \Gamma \times \mathbb{R}^3 : g^1 + g^2 + \sqrt{2} g^3 = 0 \right\}.$$

Now, we will show that the converse is true. For any $g \in C^{(1)}_{E_1,F}(\Sigma)$, we can find a smooth function $f \in C^\infty(\mathbb{R}^2)$ such that $g = f|_\Sigma$.

Note that since g^i is smooth up to the boundary, we may write

$$g^i(t) = g^i(0) + t(g^i)'(0) + t^2 h^i(t)$$

where $h^i(t)$ is defined by

$$h^i(t) = \begin{cases} \frac{g^i(t) - g^i(0) - t(g^i)'(0)}{t^2}, & t > 0, \\ \frac{(g^i)''(0)}{2}, & t = 0. \end{cases}$$

Note that $h^i(t)$ is smooth up to the boundary.

We extend $h^i(t)$ to a smooth function on $\mathbb{R}$ and define a global function $f(x, y)$ by

$$\begin{aligned} f(x,y) = g^1(0) + x(g^1)'(0) + y(g^2)'(0) + x^2 h^1(x) + y^2 h^2(y) \\ + xy \left(h^3 \left(\frac{x+y}{\sqrt{2}} \right) - h^1(x) - h^2(y) \right). \end{aligned}$$

We can easily see $f(x, y)$ is a smooth function and moreover, we can verify f is indeed an extension of g on Σ. For example, we can verify $g^3(t) = f(-\frac{t}{\sqrt{2}}, -\frac{t}{\sqrt{2}})$ by noting

$$\begin{aligned} f\left(-\frac{t}{\sqrt{2}}, -\frac{t}{\sqrt{2}}\right) &= g^3(0) - \frac{t}{\sqrt{2}} \left((g^1)'(0) + (g^2)'(0) \right) + t^2 h^3(t) \\ &= g^3(0) + t(g^3)'(0) + t^2 h^3(t) = h(t). \end{aligned}$$

In other words, we have shown, for any $g \in C^{(1)}_{E_1,F}(\Sigma)$, we can extend g to a smooth function f defined on the whole space $\mathbb{R}^2$.

This example shows that the first derivative condition is enough for the definitions of interesting spaces.

3.1.3 Sign functions related to the orientation

This subsection will define a function space related to the variation of Σ.

Since we have assumed each Σ^i is orientable, we can choose a differential n-form $\omega \in \Omega^n(\Sigma)$ such that ω^i is a non-vanishing n-form on Σ^i (hence it will determine an orientation on Σ^i) for each i. Note that each ω^i will naturally induce an orientation on Γ by the $(n-1)$-form $\iota_{\tau^i}\omega^i$. In general, for different i, the ω^i may induce a different orientation on Γ. Hence, we want to use a sign function to describe such differences.

Let us fix an orientation form η on Γ (a non-vanishing $(n-1)$-form). We define the sign function $\mathrm{sign} = (\mathrm{sign}^1, \mathrm{sign}^2, \mathrm{sign}^3)$ on Γ by

$$\mathrm{sign}^i(p) = \begin{cases} 1, & \iota_{\tau^i}\omega \text{ and } \eta \text{ have same sign at } p, \\ -1, & \iota_{\tau^i}\omega \text{ and } \eta \text{ have different signs at } p. \end{cases}$$

Now we define the smooth subbundle $F_\theta \subset \Gamma \times \mathbb{R}^3$ (related to density θ) as

$$F_\theta := \left\{ (p, (g^1, g^2, g^3)) \in \Gamma \times \mathbb{R}^3 : \sum_{i=1}^{3} \theta^i(p)\mathrm{sign}^i(p) g^i(p) = 0, \quad \forall p \in \Gamma \right\}. \tag{3.1}$$

Note that the vector bundle F_θ does not rely on the choice of the orientation on Γ. It only relies on the orientation on Σ. Hence, we can define the function space $C^{(0)}_{F_\theta}(\Sigma)$ based on F_θ.

Using the sign function, we can define the orientability of Σ.

Definition 3.5. We say a triple junction hypersurface Σ is *orientable* if we can choose orientations on each Σ^i and on Γ such that the sign function is identically 1.

Note that if Σ is orientable, then $F_\theta = E_\theta$.

Remark 3.6. *If we have a double junction hypersurface instead of the triple junction hypersurface, then the orientability defined above coincides with the usual orientability on this hypersurface.*

For example, suppose M is a smooth manifold, and it can be regarded as a union of two orientable smooth manifolds M_1, M_2 with boundary by an identification along their boundaries. Then M is orientable if and only if we can find orientations on M_1, M_2, respectively, such that they induce the same orientation on $\partial M_1 = \partial M_2$.

Note that we use the orientability in Theorem 6.7 only. Hence, this thesis also provides a method to study geometric properties and the PDE theory on non-orientable hypersurfaces if we change triple junction hypersurfaces to double junction hypersurfaces.

Now, let us illustrate why $C^{(0)}_{F_\theta}(\Sigma)$ is related to the variation.

Fix a Riemannian manifold N equipped with metric g_N. We suppose N is orientable and fix an orientation on N. Let $\phi : \Sigma \to N$ be an extrinsic triple junction hypersurface in N.

Suppose ν is a unit normal vector field on Σ. Recall that $\nu = (\nu^1, \nu^2, \nu^3)$ is a unit normal vector field if and only if each ν^i is a unit normal vector field on Σ^i for $i = 1, 2, 3$. Usually, ν will induce an orientation on Σ.

We choose $\tau = (\tau^1, \tau^2, \tau^3)$ as the unit normal vector field of Γ such that each τ^i is tangent to Σ^i and pointing outside of Σ^i. As before, we can choose an orientation on Γ. Suppose $\{e_1, \cdots, e_{n-1}\}$ is an oriented basis of $T_p\Gamma$ for any $p \in \Gamma$. Then by the definition of sign, we know $\left\{\mathrm{sign}^i \tau^i, e_1, \cdots, e_{n-1}\right\}$ is an oriented basis of Σ^i at p. Note that the orientation on Σ^i is induced by ν^i, we actually know $\left\{\nu^i, \mathrm{sign}^i \tau^i, e_1, \cdots, e_{n-1}\right\}$ is an oriented basis of N at p.

Hence, we know the vector fields $\left\{\nu^i, \mathrm{sign}^i \tau^i\right\}$ determine the same orientation on the normal bundle of Γ in N for different i and at the same time, each of them forms an orthonormal basis of E. This means, if $\sum_{i=1}^3 \theta^i \tau^i = 0$ along Γ, then we know

$$\sum_{i=1}^{3} \mathrm{sign}^i \nu^i \theta^i = 0.$$

Note that this result implies, if $\phi : \Sigma \to N$ is a minimal immersion with density θ, then for any smooth vector field X on N, we know the function f defined by $f := \langle X, \nu \rangle_{g_N}$ is in the class $C^{(0)}_{F_\theta}(\Sigma)$. Hence, this function space is closely related to the vector fields on Σ and it will appear in the first and second variation formulas.

3.2 Vector fields and differential forms

We fix a triple junction hypersurface Σ with triple junction structure (τ, D) and density $\theta = (\theta^1, \theta^2, \theta^3)$.

3.2.1 Vector fields on triple junction hypersurfaces

Definition 3.7. We say $X = (X^1, X^2, X^3)$ is a *smooth vector field* on Σ if $X^i \in \Gamma(T\Sigma^i)$ is a smooth vector field on Σ^i for each $i = 1, 2, 3$. We use $\mathfrak{X}(\Sigma)$ to denote the space of smooth vector fields on Σ.

We say $X \in \mathfrak{X}(\Sigma)$ is $C^{(1)}$ *continuously tangentially across the boundary* Γ *with density* θ, and write it as $X \in \Gamma^{(1)}_{\theta}(D)$, if in addition, the following three conditions hold,

- there exists a neighborhood U of Γ, we have $X^i_p \in D^i_p$ for all $p \in U^i$,
- $X^i|_{\Gamma} \equiv X^j|_{\Gamma}$,
- $\sum_{i=1}^{3} \theta^i [\tau^i, X^i] \equiv 0$ along Γ.

One of the properties of the vector field X in $\Gamma^{(1)}_{\theta}(D)$ is the function space $C^{(1)}_{E_1,E_\theta}(\Sigma)$ is invariant under the action of X.

Proposition 3.8. *For any* $X \in \Gamma^{(1)}_{\theta}(D)$ *and* $f \in C^{(1)}_{E_1,E_\theta}(\Sigma)$, *we have* $\frac{\partial}{\partial X} f \in C^{(1)}_{E_1,E_\theta}(\Sigma)$.

Proof. Given $f \in C^{(1)}_{E_1,E_\theta}(\Sigma)$, it is clear that $\frac{\partial}{\partial X} f \in C^{(0)}_{E_1}(\Sigma)$.

Moreover, we find
$$\frac{\partial}{\partial \tau}\left(\frac{\partial}{\partial X} f\right) = \frac{\partial}{\partial X}\left(\frac{\partial}{\partial \tau} f\right) + [\tau, X] f,$$
and note $\frac{\partial}{\partial X}\left(\frac{\partial}{\partial \tau} f\right)|_{\Gamma} \in \Gamma(E_\theta)$ and $[\tau, X] f|_{\Gamma} \in \Gamma(E_\theta)$. Hence $\frac{\partial}{\partial X} f \in C^{(1)}_{E_1,E_\theta}(\Sigma)$. $\square$

Remark 3.9. *For any subbundle* $E' \subset \Gamma \times \mathbb{R}^3$, *we can define the vector fields associated with* E' *by the following in view of Proposition 3.8.*

We define
$$\Gamma^{(1)}_{E'}(D) := \left\{ X \in \mathfrak{X}(\Sigma) : \frac{\partial f}{\partial X} \in C^{(1)}_{E_1,E'}(\Sigma), \quad \forall f \in C^{(1)}_{E_1,E'}(\Sigma) \right\}.$$

Now if we write $\mathfrak{X}_{E'}(\Gamma) := \left\{ X = (X^1, X^2, X^3) \in \oplus_{i=1}^{3} \mathfrak{X}(\Gamma) : \frac{\partial g}{\partial X} \in \Gamma(E'), \forall g \in C^\infty(\Gamma) \right\}$, *then we can find* $X \in \Gamma^{(1)}_{E'}(D)$ *if and only if*
$$X^i|_{\Gamma} \equiv X^j|_{\Gamma} \quad \textit{and} \quad [\tau, X]|_{\Gamma} \in \mathfrak{X}_{E'}(\Gamma).$$

3.2.2 Differential forms on triple junction hypersurfaces

Definition 3.10. We say $\omega = (\omega^1, \omega^2, \omega^3)$ is a *(smooth) differential form* of degree k on Σ, and write it as $\omega \in \Omega^k(\Sigma)$, if each $\omega^i \in \Omega^k(\Sigma^i)$ is a smooth differential form of degree k on Σ^i.

We say $\omega \in \Omega^k(\Sigma)$ is $C^{(1)}$ *continuous across the junction* and write it as $\omega \in \Omega^k_{E_\theta}(\Sigma)$, if in addition, we have the following conditions hold,

- $\omega(X_1, ..., X_k) \in C^{(1)}_{E_1,E_\theta}(\Sigma)$ for $X_1, \cdots, X_k \in \Gamma^{(1)}_\theta(D)$.
- $\omega(\tau, X_1, ..., X_{k-1}) \in C^{(0)}_{E_\theta}(\Sigma)$ for $X_1, \cdots, X_{k-1} \in \Gamma^{(1)}_\theta(D)$.

Recall that we use the notation $\omega(X_1, \cdots, X_k) := \left(\omega^i(X^i_1, \cdots, X^i_k)\right)^3_{i=1}$.

Clearly, any the function $f \in C^{(1)}_{E_1,E_\theta}(\Sigma)$ will automatically be a zero-form on Σ.

It is also quite interesting that such differential forms have similar properties to the usual differential forms on smooth manifolds.

Proposition 3.11. *We have the following properties for the differential forms $\Omega^*_{E_\theta}(\Sigma)$,*

- *$\Omega^*_{E_\theta}(\Sigma)$ is closed under the usual differential d on $\Omega^*(\Sigma)$.*
- *$\Omega^*_{E_\theta}$ is closed under $\wedge$ product.*

Hence, Proposition 3.11 shows $\Omega^*_{E_\theta}(\Sigma)$ is a well-defined differential graded subalgebra of $\Omega^*(\Sigma)$.

Proof. The proof is clear if we choose a suitable coordinate system. Let $(x_1, \cdots, x_n)$ be the coordinate chart near p that appeared in Proposition 2.11. Locally, we can express $\omega \in \Omega^*(\Sigma)$ as

$$\omega = a_{1I} dx_1 \wedge dx_I + a_J dx_J$$

where I, J are multi-indices that do not contain index 1. Here, we use the Einstein summation so that there is a summation over the multi-indices I, J. Note that D is spanned by $\left\{\frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right\}$ and $\tau = -\frac{\partial}{\partial x_1}$, we can find

$$a_{1I} \in C^{(0)}_{E_\theta}(\Sigma) \quad \text{and} \quad a_J \in C^{(1)}_{E_1,E_\theta}(\Sigma)$$

if and only if $\omega \in \Omega^*_{E_\theta}(\Sigma)$.

Hence for $\omega \in \Omega^*_{E_\theta}(\Sigma)$, we have

$$d\omega = \sum_{j=2}^{n} \frac{\partial a_{1I}}{\partial x_j} dx_j \wedge dx_1 \wedge dx_I + \frac{\partial a_J}{\partial x_1} dx_1 \wedge dx_J + \sum_{j=2}^{n} \frac{\partial a_J}{\partial x_j} dx_j \wedge dx_J \in \Omega^*_{E_\theta}(\Sigma)$$

by noting $\frac{\partial a_{1I}}{\partial x_j}|_\Gamma \in \Gamma(E_\theta)$, $\frac{\partial a_J}{\partial x_1}|_\Gamma \in \Gamma(E_\theta)$ and $\frac{\partial a_J}{\partial x_j} \in C^{(1)}_{E_1,E_\theta}(\Sigma)$.

So $d\omega \in \Omega^*_{E_\theta}(\Sigma)$.

The second result can be concluded using the following properties of the closeness of the function space by using the coordinate chart in Proposition 2.11. □

Proposition 3.12. *The function spaces $C^{(0)}_{E_1}(\Sigma)$ and $C^{(1)}_{E_1,E'}(\Sigma)$ are all closed under the usual multiplication for any smooth subbundle E' of $\Gamma \times \mathbb{R}^3$.*

Moreover, $C^{(0)}_{E_1}(\Sigma)$ acts on any $C^{(0)}_{E}(\Sigma)$ for any smooth k-dimensional subbundle $E \subset \Gamma \times \mathbb{R}^3$.

Proof. Indeed, $C^{(0)}_{E_1}(\Sigma)$ acts on $C^{(0)}_{E}(\Sigma)$ by multiplication comes from the definition of E_1. As a by product, we know $C^{(0)}_{E_1}(\Sigma)$ is closed under multiplication.

Now for any $f_1, f_2 \in C^{(1)}_{E_1,E'}(\Sigma)$, we know

$$\left(f_1 \frac{\partial}{\partial \tau} f_2 \right)\bigg|_\Gamma \in \Gamma(E'), \quad \left(f_2 \frac{\partial}{\partial \tau} f_1 \right)\bigg|_\Gamma \in \Gamma(E').$$

We also have $f_1 f_2 \in C^{(0)}_{E_1}(\Sigma)$. Hence $f_1 f_2 \in C^{(1)}_{E_1,E'}(\Sigma)$. □

Remark 3.13. *Note that in general, the function space $C^{(0)}_{E}(\Sigma)$ might not closed under multiplication for a general E. This is the reason why we consider $C^{(1)}_{E_1,E_\theta}(\Sigma)$ and define the differential forms only related to $C^{(1)}_{E_1,E_\theta}(\Sigma)$. Indeed, other function spaces will also play a crucial role for other kinds of elliptic PDEs.*

We can also describe the proposition of differential operators in the following sense.

Proposition 3.14. *There is a unique operator $d : \Omega^k_{E_\theta}(\Sigma) \to \Omega^{k+1}_{E_\theta}(\Sigma)$ for all k satisfying the following properties,*

- *d is linear over $\mathbb{R}$.*

- *For any $\omega \in \Omega^k_{E_\theta}(\Sigma)$ and $\eta \in \Omega^l_{E_\theta}(\Sigma)$, we have*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \eta \wedge d\eta.$$

- $d \circ d \equiv 0$.

- *For any $f \in C^{(1)}_{E_1,E_\theta}(\Sigma)$, df is given by $df(X) = \frac{\partial}{\partial X} f$ for any $X \in \mathfrak{X}(\Sigma)$.*

Proof. Suppose $\tilde{d}$ is any operator satisfying the above four conditions.

We only need to show $\tilde{d}$ is a local operator. This is because we already know the d satisfies this proposition on the usual smooth manifolds without boundary. If we know $\tilde{d}$ is a local operator up to the boundary, then $\tilde{d}$ must agree with d up to the boundary and hence $\tilde{d} = d$.

Hence, we only need to show if $\omega_1 = \omega_2$ in a neighborhood U_0 of $p \in \Gamma$, then $d\omega_1 = d\omega_2$ at p. Let us choose a bump function $\psi \in C^{(1)}_{E_1,E_\theta}(\Sigma)$ supported in U and having value 1 near p. This can be done using the coordinate chart $\varphi : U \to V$ in Proposition 2.11. For example, we can choose a bump function $\phi : \mathbb{R}^n_+ \to \mathbb{R}$ such that

$$\phi \equiv 1 \text{ in } B_\varepsilon(0) \cap \mathbb{R}^n_+, \quad \text{supp}\phi \subset B_{2\varepsilon}(0) \cap \mathbb{R}^n_+, \quad \frac{\partial}{\partial x_1}\phi = 0 \text{ along } \{x_1 = 0\}.$$

Here, we use $B_r(x)$ to denote the closed balls in Euclidean space.

Now we can choose the bump function ψ such that $\psi^i := \phi \circ \varphi^i$ for ε sufficiently small such that $\text{supp}\psi \subset U_0$. From the above construction, we know $\psi \in C^{(1)}_{E_1,E_\theta}(\Sigma)$. Note that $\psi\eta$ is identically zero with $\eta := \omega_1 - \omega_2$, hence by the properties of $\tilde{d}$, we know

$$0 \equiv \tilde{d}(\psi\eta) = \tilde{d}\psi \wedge \eta + \psi\tilde{d}\eta.$$

Note that $\tilde{d}\psi_p = 0$ by the fourth property of $\tilde{d}$ and note $\psi^i(p) = 1$, we have $\tilde{d}\eta^i = 0$ at p.

This shows $\tilde{d}\omega_1$ and $\tilde{d}\omega_2$ agrees with each other at p. Hence we have shown $\tilde{d}$ is a local operator. $\square$

3.3 Equivalence classes of triple junction hypersurfaces

As we have seen in Remark 2.12, we know we can define the function spaces and differential forms by τ only. In this section, we will see that we only need to ensure τ is defined along Γ (or just an equivalence class of τ defined along Γ).

Definition 3.15. Fix a triple junction hypersurface Σ, we say two triple junction structures (τ, D) and $(\tilde{\tau}, \tilde{D})$ are *equivalent* to each other, if the function space $C^{(1)}_{E,E'}(\Sigma)$ agrees with $\tilde{C}^{(1)}_{E,E'}(\Sigma)$ where $\tilde{C}^{(1)}_{E,E'}(\Sigma)$ is determined by $(\tilde{\tau}, \tilde{D})$ for any $E, E' \subset \Gamma \times \mathbb{R}^3$ being smooth subbundles over Γ. We will write this equivalence as $(\tau, D) \sim (\tilde{\tau}, \tilde{D})$.

This definition is quite natural. Usually, if two geometric structures give the same spaces, like function spaces, we have no reason to distinguish them when we want to solve PDEs.

Recall that in the definition of $C^{(1)}_{E,E'}(\Sigma)$, we only need the values of τ along Γ. Hence if $\tau = \tilde{\tau}$, we know $(\tau, D) \sim (\tilde{\tau}, \tilde{D})$. Moreover, for any positive function $h \in C^\infty(\Gamma)$, we have $(\tau, D) \sim (\tilde{\tau}, \tilde{D})$ if $\tau = h\tilde{\tau}$ along Γ. Conversely, we have

Proposition 3.16. *If $(\tau, D) \sim (\tilde{\tau}, \tilde{D})$, then there is a positive function $h \in C^\infty(\Gamma)$ such that $\tilde{\tau} = h\tau$.*

Proof. This is easy to see by considering the space $C^{(1)}_{\Gamma\times\mathbb{R}^3,E'}(\Sigma)$ where E' is a non-trivial subbundle. For any $p \in \Gamma$, we know $\tilde{\tau}$ can be written as $\tilde{\tau}^i = h^i\tau^i + \eta^i$ where $\eta^i \in \mathfrak{X}(\Gamma), h^i \in C^\infty(\Gamma)$. Then we may choose a test function $f \in C^{(1)}_{\Gamma\times\mathbb{R}^3,E'}(\Sigma)$ such that $\frac{\partial}{\partial\eta} f = 0$ at p for any $\eta \in T_p(\Gamma)$ (i.e., f is constant along Γ). This will imply if $(a_1, a_2, a_3) \in E'_p$, then we have $(h^1a_1, h^2a_2, h^3a_3) \in E'_p$ by $f \in \tilde{C}^{(1)}_{\Gamma\times\mathbb{R}^3,E'}(\Sigma)$. We can choose different E' to get $h^1 = h^2 = h^3$. Hence we may suppose $\tilde{\tau}^i = h\tau^i + \eta^i$ for some $h \in C^\infty(\Gamma)$ and WLOG, assume $h \equiv 1$ near p.

Hence, we get $\left(\frac{\partial f^i}{\partial\eta^i}\right)_{i=1}^3 \in \Gamma(E')$. Suppose $\eta^1 \neq 0$, then we choose $E' = \{(p, (0, g^2, g^3)) \in \Gamma \times \mathbb{R}^3\}$. For any $f \in C^{(1)}_{\Gamma\times\mathbb{R}^3,E'}(\Sigma)$, we can adjust f such that $\frac{\partial f^1}{\partial\eta^1} \neq 0$ and $\frac{\partial f^j}{\partial\eta^j} = 0$ at p for $j = 2, 3$. This is possible since we do not have restriction on f. This will contradict the fact $\left(\frac{\partial f^i}{\partial\eta^i}\right)_{i=1}^3 \in \Gamma(E')$.

Hence, we need to have $\eta^i \equiv 0$ for each i. □

It is interesting to note that the differential forms do not depend on the choice of the equivalence class of (τ, D).

Proposition 3.17. *If $(\tau, D) \sim (\tilde{\tau}, \tilde{D})$, then*

$$\Omega^*_{E_\theta}(\Sigma) = \tilde{\Omega}^*_{E_\theta}(\Sigma)$$

where $\tilde{\Omega}^*_{E_\theta}(\Sigma)$ is the space of the differential forms related to $(\tilde{\tau}, \tilde{D})$.

Proof. Using Proposition 3.16, we suppose $\tilde{\tau} = h\tau$ for $h \in C^\infty(\Sigma)$.

Suppose $\omega \in \Omega^k_{E_\theta}(\Sigma)$. Let $X_1, \cdots, X_k \in \tilde{\Gamma}^{(1)}_\theta(D)$ and let $Y_1, \cdots, Y_k \in \Gamma^{(1)}_\theta(\Sigma)$ such that $Y_j = X_j$ for $j = 1, \cdots, k$. It is easy to note

$$\omega(\tilde{\tau}, X_1, \cdots, X_{k-1}) \in C^{(0)}_{E_\theta}(\Sigma), \quad \omega(X_1, \cdots, X_k) \in C^{(0)}_{E_1}(\Sigma).$$

So we only need to show

$$\frac{\partial}{\partial \tilde{\tau}} \omega(X_1, \cdots, X_k)|_\Gamma \in \Gamma(E_\theta).$$

Note that by the properties of Lie derivative, we have

$$\mathcal{L}_{\tilde{\tau}} \omega(X_1, \cdots, X_k) = \frac{\partial}{\partial \tilde{\tau}} \omega(X_1, \cdots, X_k) - \sum_{j=1}^{k} \omega(\cdots, [\tilde{\tau}, X_j], \cdots), \tag{3.2}$$

where $\omega(\cdots, [\tilde{\tau}, X_j], \cdots)$ means $\omega(X_1, \cdots, X_{j-1}, [\tilde{\tau}, X_j], X_{j+1}, \cdots, X_k)$ and $\mathcal{L}_{\tilde{\tau}} = (\mathcal{L}^i_{\tilde{\tau}^i})^3_{i=1}$ with $\mathcal{L}^i$ the Lie derivative on Σ^i.

Since $\mathcal{L}_{\tilde{\tau}}\omega$ is a tensor, we have

$$\begin{aligned}
&\mathcal{L}_{\tilde{\tau}} \omega(X_1, \cdots, X_k)|_\Gamma = \mathcal{L}_{\tilde{\tau}} \omega(Y_1, \cdots, Y_k)|_\Gamma \\
&= h \frac{\partial}{\partial \tau} \omega(Y_1, \cdots, Y_k)|_\Gamma - \sum_{j=1}^{k} \omega(\cdots, [h\tau, Y_j], \cdots)|_\Gamma \\
&= h \frac{\partial}{\partial \tau} \omega(Y_1, \cdots, Y_k)|_\Gamma - h \sum_{j=1}^{k} \omega(\cdots, [\tau, Y_j], \cdots)|_\Gamma + \sum_{j=1}^{k} \frac{\partial h}{\partial Y_j} \omega(\cdots, \tau, \cdots)|_\Gamma
\end{aligned} \tag{3.3}$$

where we have used $\frac{\partial h}{\partial Y_j} \omega(\cdots, \tau, \cdots) := \frac{\partial h}{\partial Y_j} \omega(Y_1, \cdots, Y_{j-1}, \tau, Y_{j+1}, \cdots, Y_k)$.

Recall that $\sum_{i=1}^3 \theta^i[\tilde{\tau}^i, X_j^i] \equiv \sum_{i=1}^3 \theta^i[\tau^i, Y_j^i] \equiv 0$, we get

$$\sum_{j=1}^k \omega(\cdots, [\tau, Y_j], \cdots), \sum_{j=1}^k \omega(\cdots, [\tilde{\tau}, X_j], \cdots) \in C_{E_\theta}^{(0)}(\Sigma). \tag{3.4}$$

At last, we know $h\frac{\partial}{\partial\tau}\omega(Y_1, \cdots, Y_k)$, $\frac{\partial h}{\partial Y_j}\omega(\cdots, \tau, \cdots) \in C_{E_\theta}^{(0)}(\Sigma)$. Combining the identities (3.2), (3.3), and the result (3.4), we have $\frac{\partial}{\partial\tilde{\tau}}\omega(X_1, \cdots, X_k)|_\Gamma \in \Gamma(E_\theta)$. $\square$

From now on, we will call τ as a triple junction structure on Σ even if τ is only defined along Γ. In this case, we will extend τ arbitrarily near Γ and choose D based on τ if necessary.

If we only consider the space $C_{E_1,E_\theta}^{(1)}(\Sigma)$ when fixing the density θ on Σ, we have another type of equivalent defined as follows.

Definition 3.18. We say two triple junction structures (τ, D) and $(\tilde{\tau}, \tilde{D})$ are *θ-equivalent* to each other, if the function space $C_{E_1,E_\theta}^{(1)}(\Sigma)$ agrees with $\tilde{C}_{E_1,E_\theta}^{(1)}(\Sigma)$. We will write this equivalence as $(\tau, D) \sim_\theta (\tilde{\tau}, \tilde{D})$.

We can give another description of the θ-equivalence relation.

Proposition 3.19. *$(\tau, D) \sim_\theta (\tilde{\tau}, \tilde{D})$ if and only if there exists a positive smooth function $h \in C^\infty(\Gamma)$ and some vector field $X \in \mathfrak{X}_\theta(\Gamma)$ such that $\tilde{\tau} = h\tau + X$. Here, $\mathfrak{X}_\theta(\Gamma)$ is defined as,*

$$\mathfrak{X}_\theta(\Gamma) := \left\{ X = (X^1, X^2, X^3) : X^i \in \mathfrak{X}(\Gamma) \text{ and } \sum_{i=1}^3 \theta^i X^i \equiv 0 \right\}.$$

Proof. The "only if" part is easy to see after a simple verification.

For the "if" part, as shown in Proposition 3.16, we know $\tilde{\tau}^i = h\tau + \eta^i$ for some $h \in C^\infty(\Gamma)$ and $\eta^i \in \mathfrak{X}(\Gamma)$. Hence, the condition $f \in C_{E_1,E_\theta}^{(1)}(\Sigma)$ implies $\sum_{i=1}^3 \theta^i df^i(\eta^i) \equiv 0$ on Γ, or we can write $\sum_{i=1}^3 dg(\theta^i\eta^i) \equiv 0$ if we define $g = f|_\Gamma$. Note that for any $g \in C^\infty(\Sigma)$, we may extend it to a function $f \in C_{E_1,E_\theta}^{(1)}(\Sigma)$ with $g = f|_\Gamma$ on Γ. So the identity $\sum_{i=1}^3 dg(\theta^i\eta^i) \equiv 0$ for any g. Hence $\sum_{i=1}^3 \theta^i\eta^i \equiv 0$. $\square$

Moreover, we have a slightly stronger result than Proposition 3.17.

Proposition 3.20. *If* $(\tau, D) \sim_\theta (\tilde{\tau}, \tilde{D})$, *then*

$$\Omega^*_{E_\theta}(\Sigma) = \tilde{\Omega}^*_{E_\theta}(\Sigma).$$

Proof. In view of the result Proposition 3.17, we only need to consider the case $\tilde{\tau} = \tau + X$ for some $X \in \mathfrak{X}_\theta(\Gamma)$.

The proof for this case is similar to the proof of Proposition 3.17. The key here is to show

$$\frac{\partial}{\partial \tilde{\tau}} \omega(X_1, \cdots, X_k)|_\Gamma \in \Gamma(E_\theta),$$

and we can make use of Lie derivatives to finish the proof. □

3.4 Metrics on triple junction hypersurfaces

We say $g \in \mathrm{Sym}^2 T^*\Sigma$ is a *metric* on Σ and write it as $g \in \mathrm{Met}(\Sigma)$ if $g = (g^1, g^2, g^3)$ such that each g^i is a metric on Σ^i for each $i = 1, 2, 3$.

3.4.1 Canonical metrics and $C^{(1)}$ metrics

Given (τ, D), the triple junction structure on Σ, we can define a canonical metric as follows.

Definition 3.21. We say $g \in \mathrm{Met}(\Sigma)$ is a *canonical metric* on Σ if the following holds,

- $g = (g^1, g^2, g^3)$ such that $g^1|_\Gamma = g^2|_\Gamma = g^3|_\Gamma$ on Γ,
- $g(\tau, \tau) \equiv 1$,
- $g(\tau, X) \equiv 0$ for any $X \in \Gamma(D)$,
- $g(X, Y) \in C^{(1)}_{E_1, E_\theta}(\Sigma)$ for any $X, Y \in \Gamma^{(1)}_\theta(D)$.

Note that the canonical metric will depend on D and it is not unique. Recall that if we use the coordinate chart in Proposition 2.11, we know the canonical metric g has the form

$g = dx_1^2 + g_{x_1}$ where g_t is the metric on $\{x_1 = t\}$. This expression shows that the canonical metric has some relations with the Fermi coordinate. We will use this fact later on.

Let us define a $C^{(1)}$ metric on Σ, which only relies on the equivalence class of triple junction structures.

Definition 3.22. We say a metric $g \in \mathrm{Met}(\Sigma)$ is a $C^{(1)}$ *metric* on Σ, and write it as $g \in \mathrm{Met}^{(1)}(\Sigma)$ if g satisfies the following conditions,

- $g = (g^1, g^2, g^3)$ such that $g^1|_\Gamma = g^2|_\Gamma = g^3|_\Gamma$ on Γ,
- $g(\tau^\perp, \tau^\perp) \in C^{(0)}_{E_1}(\Sigma)$,
- $g(\tau, X) \in C^{(0)}_{E_\theta}(\Sigma), \forall X \in \mathfrak{X}(\Sigma)$ along Γ,
- $g(X, Y) \in C^{(1)}_{E_1, E_\theta}(\Sigma), \forall X, Y \in \Gamma^{(1)}_\theta(D)$.

Here, $\tau^\perp$ means the projection of τ to the normal of D under metric g.

By the way, we write $g \in \mathrm{Met}^{(0)}(\Sigma)$ if we only have $g^i|_\Gamma \equiv g^j|_\Gamma$ on Γ.

Remark 3.23. *We note each canonical metric* $g \in \mathrm{Met}^{(1)}(\Sigma)$.

We may prove the following proposition.

Proposition 3.24. *The definition of $C^{(1)}$ metric does not rely on the choice of elements in the equivalence class of relation $\sim_\theta$.*

Proof. Suppose $(\tau, D) \sim_\theta (\tilde{\tau}, \tilde{D})$ and $g \in \mathrm{Met}^{(1)}(\Sigma)$. Then we want to show $g \in \tilde{\mathrm{Met}}^{(1)}(\Sigma)$ where the $\tilde{\mathrm{Met}}^{(1)}(\Sigma)$ is the space of $C^{(1)}$ metric related to $(\tilde{\tau}, \tilde{D})$.

From Proposition 3.19, we know $\tilde{\tau} = h\tau + \eta$ for some positive $h \in C^\infty(\Gamma)$ and $\eta \in \mathfrak{X}_\theta(\Gamma)$. Suppose $\tilde{X}, \tilde{Y} \in \tilde{\Gamma}^{(1)}_\theta(D)$ and $X, Y \in \Gamma^{(1)}_\theta(D)$ such that $X|_\Gamma = \tilde{X}|_\Gamma, Y|_\Gamma = \tilde{Y}|_\Gamma$.

It is easy to verify the following things,

$$\begin{aligned} g(\tilde{\tau}^\perp, \tilde{\tau}^\perp)|_\Gamma &= h^2 g(\tau^\perp, \tau^\perp)|_\Gamma \in \Gamma(E_1), \\ g(\tilde{\tau}, \tilde{X})|_\Gamma &= hg(\tau, X)|_\Gamma + g(\eta, X)|_\Gamma \in \Gamma(E_\theta), \\ g(\tilde{X}, \tilde{Y})|_\Gamma &= g(X, Y)|_\Gamma \in \Gamma(E_1). \end{aligned}$$

Hence, we only need to show,

$$\frac{\partial}{\partial \tilde{\tau}} g(\tilde{X}, \tilde{Y})|_{\Gamma} \in \Gamma(E_\theta).$$

This can be done by using the properties of Lie derivatives. Note that along Γ, we have

$$\begin{aligned}
&\frac{\partial}{\partial \tilde{\tau}} g(\tilde{X}, \tilde{Y}) - g([\tilde{\tau}, \tilde{X}], \tilde{Y}) - g(\tilde{X}, [\tilde{\tau}, \tilde{Y}]) \\
&= \mathcal{L}_{\tilde{\tau}} g(\tilde{X}, \tilde{Y}) = \mathcal{L}_{\tilde{\tau}} g(X, Y) \\
&= h\frac{\partial}{\partial \tau} g(X, Y) + \frac{\partial}{\partial \eta} g(X, Y) - g([\tilde{\tau}, X], Y) - g(X, [\tilde{\tau}, Y]) \\
&= h\frac{\partial}{\partial \tau} g(X, Y) + \frac{\partial}{\partial \eta} g(X, Y) - g([\tau, X], Y) - g(X, [\tau, Y]) + \frac{\partial h}{\partial X} g(\tau, Y) + \frac{\partial h}{\partial Y} g(X, \tau) \\
&\quad - g([\eta, X], Y) - g(X, [\eta, Y])
\end{aligned} \tag{3.5}$$

Note that all the terms in the last part of identity (3.5) is in $C^{(0)}_{E_\theta}(\Sigma)$ by the definition of g, the properties of η, X and Y.

So we get $\frac{\partial}{\partial \tilde{\tau}} g(\tilde{X}, \tilde{Y})|_{\Gamma} \in \Gamma(E_\theta)$ by showing $g([\tilde{\tau}, \tilde{X}], \tilde{Y}), g(\tilde{X}, [\tilde{\tau}, \tilde{Y}]) \in C^{(0)}_{E_\theta}(\Sigma)$. □

Although our $C^{(1)}$ metric only relies on the equivalence classes, we can always choose a triple junction structure in its equivalence class to turn the metric into the canonical metric. This property can be viewed as the inverse property of Remark 3.23.

Proposition 3.25. *Suppose $g \in \mathrm{Met}^{(1)}(\Sigma)$, then we can find $(\tilde{\tau}, \tilde{D}) \sim_\theta (\tau, D)$ such that g is a canonical metric under the triple junction structure $(\tilde{\tau}, \tilde{D})$.*

Proof. For any $p \in \Gamma$, we fix a local coordinate chart $\varphi_\Gamma : U_\Gamma \cap \Gamma \to V_\Gamma$ of Γ near p where U_Γ is a neighborhood of p in Γ and V_Γ is a neighborhood of 0 in $\mathbb{R}^{n-1} = \left\{(x_2, \cdots, x_n) \in \mathbb{R}^{n-1}\right\}$. Now we can extend φ_Γ to a Fermi coordinate of Σ^i near p, and we write it as

$$\varphi^i : U^i \cap \Sigma^i \to V_\Gamma \times [0, \varepsilon)$$

for ε small enough.

This will give us the construction of $\tilde{\tau}$ and $\tilde{D}$ as we can choose $\tilde{\tau} = -\frac{\partial}{\partial x_1}$, $\tilde{D} = \tau^{\perp_g}$ where $\tau^{\perp_g}$ is the orthogonal complemented subspace with respect to τ under metric g. Then we can glue $\tilde{\tau}, \tilde{D}$ near Γ locally by the partition of unity. Clearly, we know g satisfies the first three conditions in the definition of the canonical metric.

Now, let us verify $(\tau, D) \sim_\theta (\tilde{\tau}, \tilde{D})$. We write $\tilde{\tau} = h\tau + \eta$ for $0 < h \in \oplus_{i=1}^3 C^\infty(\Gamma), \eta \in \oplus_{i=1}^3 \mathfrak{X}(\Gamma)$. We extend η near Γ so that $\eta \in \Gamma_\theta^{(1)}(\Sigma)$. We only need to show $h \in \Gamma(E_1)$ and $\eta \in \mathfrak{X}_\theta(\Gamma)$.

Indeed, from the definition of $\mathrm{Met}^{(1)}(\Sigma)$ and $\tilde{\tau} = \tilde{\tau}^\perp = h\tau^\perp + \eta^\perp = h\tau^\perp$ along Γ, we have

$$\frac{1}{h^2} = \frac{1}{h^2} g(\tilde{\tau}, \tilde{\tau}) = g(\tau^\perp, \tau^\perp) \in \Gamma(E_1) \quad \text{along } \Gamma.$$

Hence $h \in \Gamma(E_1)$. So we can view h as a function in $C^\infty(\Sigma)$.

On the other hand, for any $X \in \mathfrak{X}(\Gamma)$, we have $g(\tau, X)|_\Gamma \in \Gamma(E_\theta)$. Hence

$$\begin{aligned} 0 &= \sum_{i=1}^3 \theta^i g^i(\tau^i, X) = \sum_{i=1}^3 \theta^i g^i\left(\frac{\tilde{\tau}^i - \eta^i}{h}, X\right) = -\sum_{i=1}^3 \frac{\theta^i}{h} g^i(\eta^i, X) \\ &= -\sum_{i=1}^3 \frac{1}{h} g_\Gamma(\theta^i \eta^i, X) \end{aligned}$$

holds along Γ. Hence $\sum_{i=1}^3 \theta^i \eta^i \equiv 0$ by the arbitrariness of X. Hence, we know $(\tau, D) \sim_\theta (\tilde{\tau}, \tilde{D})$.

Using Proposition 3.19, we know g also satisfies the fourth condition in the definition of canonical metrics.

Hence, g is a canonical metric with triple junction structure $(\tilde{\tau}, \tilde{D})$. □

3.4.2 Geometric meanings of $C^{(1)}$ metrics

From the definition of $C^{(1)}$ metrics, we know not all metrics $g = (g^1, g^2, g^3) \in \mathrm{Met}(\Sigma)$ can become a $C^{(1)}$ metric for some triple junction structure (τ, D). At least, when we fix a $C^{(0)}$ triple junction hypersurface Σ, we need to ensure $g^i = g^j$ along Γ at least.

Indeed, we need additional conditions on g to ensure we can find (τ, D) such that g will become a $C^{(1)}$ metric under this triple junction structure. As we will see, this can be viewed as another definition of the triple junction structure.

Proposition 3.26. *Let us fix a triple junction structure (τ, D), and a metric $g \in \mathrm{Met}^{(1)}(\Sigma)$. We* *write A^i_Γ as the second fundamental form of Γ in Σ^i with respect to the outer unit normal vector field along Γ. We write $A_\Gamma = (A^1_\Gamma, A^2_\Gamma, A^3_\Gamma)$ for short. Then we have*

$$\sum_{i=1}^{3} \theta^i A^i_\Gamma(X,Y) = 0, \quad \forall X,Y \in \mathfrak{X}(\Gamma).$$

Proof. WLOG (using Proposition 3.25), we assume g is a canonical metric. Hence τ will be the outer unit normal vector field along Γ.

Hence, for any $X,Y \in \mathfrak{X}(\Gamma)$, we extend them to vector fields $X,Y \in \Gamma^{(1)}_\theta(D)$. Then we find

$$\begin{aligned}
\frac{\partial}{\partial \tau} g(X,Y) &= g(\nabla_\tau X, Y) + g(X, \nabla_\tau Y) \\
&= g([\tau, X], Y) + g(\nabla_X \tau, Y) + g(X, [\tau, Y]) + g(X, \nabla_Y \tau) \\
&= g([\tau, X], Y) + g(X, [\tau, Y]) - 2A_\Gamma(X,Y) \qquad (3.6)
\end{aligned}$$

along Γ. Recall that $[\tau, X], [\tau, Y] \in \mathfrak{X}_\theta(\Gamma)$ and $\frac{\partial}{\partial \tau} g(X,Y)|_\Gamma \in \Gamma(E_\theta)$, we get $A_\Gamma(X,Y) \in \Gamma(E_\theta)$. This is what we want to prove. □

Moreover, the reverse part is true. We can always make it into a $C^{(1)}$ metric if in addition, we have $A_\Gamma(X,Y) \in \Gamma(E_\theta)$. We summarize it as follows.

Proposition 3.27. *If $g \in \mathrm{Met}(\Sigma)$ satisfies*

- $g(X,Y) \in \Gamma(E_1), \quad \forall X,Y \in \mathfrak{X}(\Gamma)$,
- $A_\Gamma(X,Y) \in \Gamma(E_\theta), \quad \forall X,Y \in \mathfrak{X}(\Gamma)$.

Then there is a triple junction structure (τ, D) such that g is a $C^{(1)}$ metric under (τ, D).

Proof. We can construct (τ, D) as in the proof of Proposition 3.25 using Fermi coordinate. Clearly, g will satisfy the first three conditions in the definition of canonical metrics. We need a different proof to show that the fourth condition is true.

Indeed, this is not hard by noting the identity (3.6) is true and we already know $A_\Gamma(X,Y) \in \Gamma(E_\theta)$. This will give us $\frac{\partial}{\partial \tau} g(X,Y)|_\Gamma \in \Gamma(E_\theta)$. Here, we have extended X,Y such that $X, Y \in \Gamma^{(1)}_\theta(D)$. Hence, we see g is indeed a canonical metric under the triple junction structure (τ, D). $\square$

Proposition 3.27 shows that, given a $C^{(0)}$ triple junction hypersurface Σ, we can construct a $C^{(1)}$ metric first based on the conditions in Proposition 3.27, then we can recover the triple junction structures. This means $C^{(1)}$ metric itself can be viewed as a triple junction structure in some sense. Hence, we can call a metric $C^{(1)}$ even without the triple junction structure.

Remark 3.28. *We can view the conditions in Proposition 3.27 for g as the $C^{(1)}$-continuity of metrics if we recall the definition of $C^{(1)}$-continuity function space $C^{(1)}_{E_1,E_\theta}(\Sigma)$.*

In particular, recall that $g = dx_1^2 + g_{x_1}$ if we use the Fermi coordinate. Then the first condition is equivalent to $g^i_0 = g|_\Gamma$, and the second condition is equivalent to $\sum_{i=1}^3 \theta^i \frac{\partial}{\partial x_1} g^i_{x_1}|_{x_1=0} = 0$.

Remark 3.29. *Sometimes, we may also be interested in the following problem. Given $g \in \mathrm{Met}(\Sigma)$, can we find a $C^{(0)}$ triple junction structure on Σ to ensure g is a $C^{(1)}$ metric? In other words, we allow the diffeomorphisms on the boundary to change. This is the case considered in [Wan21b]. We will go back to this case later on.*

Remark 3.30. *Recall that we can define the vector fields $\Gamma^{(1)}_{E'}(D)$ associated with vector bundle E' in Remark 3.9. Indeed, we can define the differential forms and $C^{(1)}$ metrics with respect to E' using similar methods. All the results hold for this general case. Almost all the proofs are essentially the same, and we do not repeat them here since we do not use these spaces later on.*

3.4.3 Relations with extrinsic metrics

Fix (N, g_N) to be an $(n+1)$-dimensional Riemannian manifold. We let $\phi : \Sigma \to N$ be a smooth minimal immersion with density θ defined in Subsection 3.1.1.

We write the metric on the ambient manifold N as g_N. Clearly, Σ will automatically have the pullback metric $g := \phi^*(g_N)$. We will show that g will be a $C^{(1)}$ metric on Σ with density θ.

Proposition 3.31. *The pullback metric g will satisfy the conditions in Proposition 3.27. Hence, it* *will be a $C^{(1)}$ metric under some triple junction structure (τ, D). Moreover, we can choose τ such that τ^i is the unit outer normal of Γ in Σ^i.*

Proof. Note that $g^i|_\Gamma = g^j|_\Gamma$ for $i, j = 1, 2, 3$ since g is the pullback metric. So we only need to show $\sum_{i=1}^3 \theta^i A^i_\Gamma \equiv 0$.

We construct (τ, D) as in the proof of Proposition 3.25. If g satisfies the conditions in Proposition 3.27, then g will automatically be a $C^{(1)}$ metric under (τ, D).

Note that τ^i will be the unit outer normal of Γ in Σ^i, so the minimality condition implies $\sum_{i=1}^3 \theta^i \tau^i = 0$. Hence

$$\begin{aligned}\sum_{i=1}^3 \theta^i A^i_\Gamma(X, Y) &= \sum_{i=1}^3 \theta^i g^i(\nabla^{g^i}_X Y, \tau^i) = \sum_{i=1}^3 \theta^i g_N(\nabla^{g^i}_X Y, \tau^i) = \sum_{i=1}^3 \theta^i g_N(\nabla^{g_N}_X Y, \tau^i) \\ &= \sum_{i=1}^3 g_N(\nabla^{g_N}_X Y, \theta^i \tau^i) = 0\end{aligned}$$

for any $X, Y \in \mathfrak{X}(\Gamma)$, where ∇^g is the covariant derivative related to metric g. $\square$

Proposition 3.31 shows that minimal immersion is closely related to the triple junction hypersurfaces. This is one of the reasons why we impose the triple junction structure on Σ. This structure will help us study the properties of triple junction hypersurfaces arising from minimal immersions even without the metrics.

3.4.4 Conformal invariance of $C^{(1)}$ metrics

In particular, we note the $\mathrm{Met}^{(1)}(\Sigma)$ is invariant under conformal transformations in the following sense.

Proposition 3.32. *If $g \in \mathrm{Met}^{(1)}(\Sigma)$, then for any $h \in C^{(1)}_{E_1,E_\theta}(\Sigma)$ with $h > 0$, we have $hg \in \mathrm{Met}^{(1)}(\Sigma)$.*

Proof. The only non-trivial thing we need to verify $hg \in \mathrm{Met}^{(1)}(\Sigma)$ is to show

$$hg(X, Y) \in C^{(1)}_{E_1,E_\theta}(\Sigma), \quad \forall X, Y \in \Gamma^{(1)}_\theta(D).$$

But this is not hard by noting $h, g(X, Y) \in C^{(1)}_{E_1,E_\theta}(\Sigma)$ and we can use Proposition 3.12. $\square$

3.5 Diffeomorphisms between triple junction hypersurfaces

Let (Σ,Γ), $\left(\tilde{\Sigma},\tilde{\Gamma}\right)$ be two triple junction hypersurfaces. Let us fix the triple junction structure (τ, D) (resp. $(\tilde{\tau},\tilde{D})$) on Σ (resp. $\tilde{\Sigma}$). We suppose these two triple junction hypersurfaces carry the same density function θ.

Definition 3.33. We say $F = (F^1, F^2, F^3)$ is a diffeomorphism between Σ and $\tilde{\Sigma}$ if each F^i is a diffeomorphism between Σ^i and $\tilde{\Sigma}^i$ and when we restrict F^i on Γ^i or on $\mathring{\Gamma}^i$, it is a diffeomorphism.

We say F is a $C^{(0)}$ diffeomorphism between Σ and $\tilde{\Sigma}$ if F is a diffeomorphism and $F^i|_\Gamma = F^j|_\Gamma$. (Keep the $C^{(0)}$ triple junction structure.)

We say F is a $C^{(1)}$ diffeomorphism between Σ and $\tilde{\Sigma}$ if F is a $C^{(0)}$ diffeomorphism and there is $(\hat{\tau},\hat{D}) \sim_\theta (\tilde{\tau},\tilde{D})$ such that $F_*(\tau) = \hat{\tau}$ along Γ. (Keep the $C^{(1)}$ triple junction structure.)

Note that if F is a $C^{(0)}$ diffeomorphism, the pullback of function space $C^{(0)}_E(\tilde{\Sigma})$ will be the same with the function space $E^{(0)}_{F_*E}(\Sigma)$ where F_*E is the pullback bundle of E on Γ.

If F is a $C^{(1)}$ diffeomorphism, we know the pullback of $C^{(1)}_{E_1,E_\theta}(\tilde{\Sigma})$ will be $C^{(1)}_{E_1,E_\theta}(\Sigma)$.

Usually, we will always assume there is a metric on Σ, so we may not care about the diffeomorphism. However, when we want to study the properties less related to metrics, like conformal structure, we may need to consider the diffeomorphism. We will talk about the conformal structure on triple junction surfaces later on.

Chapter 4

Sobolev spaces and elliptic operators on triple junction hypersurfaces

In this chapter, we will fix a triple junction hypersurface Σ with a triple junction structure (τ, D). We will assume Σ is compact. That is, we assume each Σ^i is compact. Suppose Σ carries a density θ as usual. We will also fix a $C^{(1)}$ metric g on Σ. WLOG, we assume g is a canonical metric. Under these conditions, we know the gradient $\nabla := \nabla^g$ is well-defined on Σ.

4.1 Sobolev spaces on triple junction hypersurfaces

Definition 4.1. For any $1 \le p < \infty$, we define the L^p space on Σ by letting

$$L^p(\Sigma) := \bigoplus_{i=1}^{3} L^p(\Sigma^i).$$

Similarly, we can define the $W^{k,p}(\Sigma)$ space on Σ by letting

$$W^{k,p}(\Sigma) := \bigoplus_{i=1}^{3} W^{k,p}(\Sigma^i) \text{ for } k \ge 1.$$

Roughly speaking, for $u = (u^1, u^2, u^3) \in W^{k,p}(\Sigma)$ (resp. $u \in L^p(\Sigma)$), we simply mean each $u^i \in W^{k,p}(\Sigma^i)$ (resp. $u^i \in L^p(\Sigma^i)$).

Remark 4.2. *Since we have assumed Σ is compact, we know $L^p_{\mathrm{loc}}(\Sigma^i) = L^p(\Sigma^i)$ and $W^{k,p}_{\mathrm{loc}}(\Sigma^i) = W^{k,p}_{\mathrm{loc}}(\Sigma^i)$.*

Note that in Definition 4.1, the spaces do not even rely on the $C^{(0)}$ triple junction structure on Σ. Indeed, we know there is no meaning to talk about the restriction of the L^p function on Γ. We cannot impose the boundary condition for the functions in $L^p(\Sigma)$ using $C^{(0)}$ triple junction structure.

However, we can indeed talk about the restriction of $W^{k,p}$ functions on the boundary.

Definition 4.3. We suppose $E \subset \Gamma \times \mathbb{R}^3$ is a smooth subbundle of $\Gamma \times \mathbb{R}^3$. Then we define $W^{k,p}$ spaces related to E by

$$W^{k,p}_E(\Sigma) := \left\{ u \in W^{k,p}(\Sigma) : u|_\Gamma \in \Gamma(E) \right\},$$

where we understand the restriction in the trace sense.

Here, we have used the notation $u|_\Gamma \in \Gamma(E)$ to mean u is a section of E defined almost everywhere with respect to the measure on Γ induced by the metric $g|_\Gamma$.

In particular, we can show that $W^{k,p}_E(\Sigma)$ is a closed subset of $W^{k,p}(\Sigma)$ under the norm $\|\cdot\|_{W^{k,p}(\Sigma)}$ as follows.

Proposition 4.4. *$W^{k,p}_E(\Sigma)$ is a closed subset of $W^{k,p}(\Sigma)$ under the norm $\|\cdot\|_{W^{k,p}(\Sigma)}$. Hence, $W^{k,p}_E(\Sigma)$ is a Banach space.*

Proof. Let $\{u_j\} \subset W^{k,p}_E(\Sigma)$ be a sequence of functions such that $u_j \to u$ for some $u \in W^{k,p}(\Sigma)$ in the sense of $W^{k,p}$ norm.

Now let us fix $p \in \Gamma$ and we want to show $u|_\Gamma$ is a section of E near p. Since E is a smooth vector bundle, we know E is spanned by some orthonormal vector-valued functions $g_1, \cdots, g_l$ defined near p on Γ. (Recall that each g_s is a vector-valued function, i.e., $g_s = (g_s^1, g_s^2, g_s^3)$.) So from $u_j \in W^{k,p}_E(\Sigma)$, we know $u_j|_\Gamma$ can be written as the linear combination of g_s such that the coefficients are in L^p space near p. That is, we have

$$u_j|_\Gamma = \sum_{s=1}^{l} \tilde{u}_{js} g_s, \quad \text{for some } \tilde{u}_{js} \in L^p(\Gamma) \text{ near p.}$$

Note that $\tilde{u}_{js}$ can be obtained by the inner product with g_s as $\tilde{u}_{js} = \sum_{i=1}^{3} u^i|_\Gamma g_s^i$.

We can extend $\{g_1, \cdots, g_l\}$ to the smooth orthonormal frame $\{g_1, \cdots, g_3\}$ for $\Gamma \times \mathbb{R}^3$ near p. So we may also write $u|_\Gamma = \sum_{s=1}^{3} \tilde{u}_s g_s$ since $u \in W^{k,p}(\Sigma)$.

The trace theorem says, if we have $u_j \to u$ in the $W^{1,p}$ norm, then the restriction $u_j|_\Gamma \to u|_\Gamma$ convergences at least in L^p norm. So we know $\tilde{u}_{js} \to \tilde{u}_s$ for $s \leq l$ and $0 \to \tilde{u}_s$ for $s \geq l+1$ in the L^p sense near p. This implies $u|_\Gamma \in \Gamma(E)$ locally near p. Since Γ is compact, we know $u \in W_E^{k,p}(\Sigma)$. $\square$

Remark 4.5. *Here we will suppose E is a smooth subbundle of* $\Gamma \times \mathbb{R}^3$*, although we may consider some weaker continuity E to extend our definitions. This is because the smooth case is enough for our later application, and we can avoid some tedious technical definitions if E is smooth.*

Note that the definition of $W_E^{k,p}(\Sigma)$ does not require g to be a $C^{(1)}$ metric. In other words, Definition 4.3 does not rely on the triple junction structure on Σ. It only relies on the $C^{(0)}$ triple junction structure on Σ.

There are two spaces we use often. We define the following spaces,

$$\begin{aligned} W_1^{k,p}(\Sigma) &:= W_{E_1}^{k,p}(\Sigma), \quad W_\theta^{k,p}(\Sigma) := W_{F_\theta}^{k,p}(\Sigma). \\ H_1^k(\Sigma) &:= W_1^{k,2}(\Sigma), \qquad H_\theta^k(\Sigma) := W_\theta^{k,2}(\Sigma). \end{aligned}$$

Sometimes, we will use another equivalent norm $\|\cdot\|_{W_\theta^{k,p}(\Sigma)}$ instead of $\|\cdot\|_{W^{k,p}(\Sigma)}$ on $W_\theta^{k,p}(\Sigma)$ defined by

$$\|u\|_{W_\theta^{k,p}(\Sigma)} := \left[\sum_{i=1}^{3} \theta^i \|u^i\|_{W^{k,p}(\Sigma^i)}^p\right]^{\frac{1}{p}}.$$

Similarly, we define L_θ^p norm by

$$\|u\|_{L_\theta^p(\Sigma)} := \left[\sum_{i=1}^{3} \theta^i \int_{\Sigma^i} |u|^p \, d\Sigma^i\right]^{\frac{1}{p}}.$$

Note that if $p = 2$, then $W_E^{k,2}(\Sigma)$ will become a Hilbert space under the inner product $(\cdot,\cdot)_{W^{k,2}(\Sigma)}$ (or the inner product $(\cdot,\cdot)_{W_\theta^{k,2}(\Sigma)}$). We write $H_E^k(\Sigma) := W^{k,2}(\Sigma)$ and have the following result by Proposition 4.4.

Proposition 4.6. *$H^k_E(\Sigma)$ is a closed subset of $H^k(\Sigma)$. Moreover, $H^k_E(\Sigma)$ is a Hilbert space with inner product $(\cdot,\cdot)_{H^k(\Sigma)}$ (or $(\cdot,\cdot)_{H^k_\theta(\Sigma)}$) restricted on $H^k_E(\Sigma)$.*

4.1.1 Trace-zero spaces

We define the trace-zero space on Σ as

$$W^{k,p}_0(\Sigma) := \left\{ u \in W^{k,p}(\Sigma) : u^i|_{\Gamma^i} = 0, \quad \forall i = 1,2,3 \right\}, \quad H^k_0(\Sigma) := W^{k,2}_0(\Sigma).$$

Here, we understand $u^i|_\Gamma$ in the trace sense and $u^i|_\Gamma = 0$ means $u^i|_\Gamma$ is zero almost everywhere.

Hence, we can define the trace-zero spaces related to a smooth subbundle $E \subset \Gamma \times \mathbb{R}^3$ as

$$W^{k,p}_{0,E}(\Sigma) := W^{k,p}_0(\Sigma) \cap W^{k,p}_E(\Sigma).$$

Note that $W^{k,p}_{0,E}(\Sigma)$ is a closed subspace of $W^{k,p}(\Sigma)$ under the norm $\|\cdot\|_{W^{k,p}(\Sigma)}$. In particular, we can find the space is the closure of the following set

$$\left\{ u \in C^{(0)}_E(\Sigma) : u \text{ vanishes near } \partial\Sigma \right\}$$

under the norm $\|\cdot\|_{W^{k,p}(\Sigma)}$.

Similarly, we can define the trace-zero Hilbert spaces $H^k_{0,E}(\Sigma) := W^{k,2}_0(\Sigma) \cap W^{k,2}_E(\Sigma)$. We also use the following notations.

- $W^{k,p}_{0,1}(\Sigma) := W^{k,p}_1(\Sigma) \cap W^{k,p}_0(\Sigma)$, $W^{k,p}_{0,\theta}(\Sigma) := W^{k,p}_\theta(\Sigma) \cap W^{k,p}_0(\Sigma)$.
- $H^k_{0,1}(\Sigma) = W^{k,2}_{0,1}(\Sigma), H^k_{0,\theta}(\Sigma) = W^{k,2}_{0,\theta}(\Sigma)$.

4.2 Some integral notations

We make the integral convention to simplify our equations. Suppose Σ is a triple junction hypersurface with $C^{(1)}$ metric g.

For some function $f = (f^1, f^2, f^3)$ defined on Σ, we write

$$\int_\Sigma f d\Sigma := \sum_{i=1}^{3} \int_{\Sigma^i} f^i d\Sigma^i_{g^i}.$$

Hence, the L^p_θ norm of f on Σ can be written as $\|f\|^p_{L^p_\theta(\Sigma)} = \int_\Sigma \theta \, |f|^p \, d\Sigma$ (recall that we have defined $|f|^p := (|f^i|^p)_{i=1}^3$).

Similarly, for any function $f = (f^1, f^2, f^3)$ defined on Γ, we will write

$$\int_\Gamma f d\Gamma := \sum_{i=1}^{3} \int_\Gamma f^i d\Gamma_g.$$

4.3 Second-order elliptic operators

We recall some basic definitions and results for elliptic operators on manifolds. Note that these definitions do not rely on the $C^{(0)}$ triple junction structure on Σ. Hence we may forget about any triple junction structures on Σ (only put metric g on Σ). Hence, the following results are pretty standard, and we will not give detailed proof here. Readers may find them in some well-known textbooks (for instance, [Nic20, Chapter 10]).

4.3.1 Definitions from the algebraic aspect

The operators $\mathcal{OP}(\Sigma)$ on Σ is the space of all the $\mathbb{R}$-linear operators on $C^\infty(\Sigma) \to C^\infty(\Sigma)$. We define the adjoint map $ad(f) : \mathcal{OP}(\Sigma) \to \mathcal{OP}(\Sigma)$ by $\mathrm{ad}(f)(T) := T \circ f - f \circ T = [T, f]$ for any $T \in \mathcal{OP}(\Sigma)$ and $f \in C^\infty(\Sigma)$.

Now, we define $\mathcal{P}^{(m)}(\Sigma)$ as *partial differential operators* of order $\leq m$ by

$$\mathcal{P}^{(m)}(\Sigma) := \left\{ T \in \mathcal{OP}(\Sigma) : [T, f] \in \mathcal{P}^{(m-1)}, \quad \forall f \in C^\infty(\Sigma) \right\}, m \geq 1,$$

and we write $P^{(0)}(\Sigma) := \{T \in \mathcal{OP}(\Sigma) : [T, f] = 0, \forall f \in C^\infty(\Sigma)\}$. Note that $\mathcal{P}^{(0)}(\Sigma)$ can be identified with $C^\infty(\Sigma)$ by multiplication.

Lemma 4.7. *For any $P \in \mathcal{P}^{(m)}(\Sigma)$, we have $\mathrm{ad}(f) \circ \mathrm{ad}(g)(P) = \mathrm{ad}(g) \circ \mathrm{ad}(f)(P)$ for any $f, g \in C^\infty(\Sigma)$. Moreover, if $f_i, g_i \in C^\infty(\Sigma)$ such that at $q \in \Sigma$, we have $df_i(q) = dg_i(q)$ for*

$i = 1, \cdots, m$, *then*

$$[\mathrm{ad}(f_1)\mathrm{ad}(f_2)\cdots\mathrm{ad}(f_m)P]\,|_q = [\mathrm{ad}(g_1)\mathrm{ad}(g_2)\cdots\mathrm{ad}(g_m)P]\,|_q.$$

Lemma 4.7 tells us, the value of $\frac{1}{m!}\,[\mathrm{ad}(f_1)\mathrm{ad}(f_2)\cdots\mathrm{ad}(f_m)P]\,|_p$ depends only on covector $\xi_i = df_i(q) \in T_q^*\Sigma$. Hence, we may define the linear function on $T_q^{*\otimes^m}M$ by

$$\sigma(P)(\xi_1, \cdots, \xi_m) = \frac{1}{m!}\,[\mathrm{ad}(\xi_1)\cdots\mathrm{ad}(\xi_m)]\,P|_q.$$

Since $\sigma(P)$ is symmetric in the variables ξ_i by Lemma 4.7, we know the linear function $\sigma(P)$ is uniquely determined by the degree m polynomial $\sigma_m(P)(\xi) := \sigma(P)(\xi, \cdots, \xi)$ (might be a zero polynomial).

We say $P \in \mathcal{P}^{(m)}(\Sigma)$ has order m if $\sigma_m(P)$ is not identically to zero. We will call $\sigma_m(P)$ as the principal symbol of P. In particular, we say P is a *second-order operator* on Σ if $m = 2$.

We say a second-order operator P is elliptic if for any $q \in \Sigma$ and $\xi \in T_q^*(\Sigma)\setminus\{0\}$, we have $\sigma_2(P)(\xi) > 0$.

Remark 4.8. *In general, the ellipticity of a general degree operator only requires $\sigma_m(P)(\xi) \neq 0$. Since we are only interested in second-order operators that are positive or negative defined, we will assume $\sigma_2(P)(\xi) > 0$ here.*

Remark 4.9. *We do not need the metric g to define the partial differential operators from the above definitions.*

Since we have assumed Σ is compact, we know if P is an elliptic second-order operator, then P is uniformly elliptic in the sense that $\frac{1}{\Lambda}\,|\xi|^2 \leq \sigma_2(P) \leq \Lambda\,|\xi|^2$ for some Λ large enough.

For any $P \in \mathcal{P}^{(m)}(\Sigma)$, we say a operator $Q \in \mathcal{P}^{(m)}(\Sigma)$ is a formal adjoint of P, written as $P^* := Q$, if for any $u, v \in C_0^\infty(\Sigma)$, we have

$$\int_\Sigma vP(u)d\Sigma = \int_\Sigma uQ(v)d\Sigma.$$

Here, $C_0^\infty(\Sigma)$ means the functions that vanish near $\partial\Sigma^i$ on each Σ^i. We say P is formally self-adjoint if $P^* = P$.

Remark 4.10. *Note that we need the metric g here to define the adjoint-operator P^*. This is because the volume form $d\Sigma$ is associated with g.*

If P is an elliptic second-order operator, then in the local coordinate $\{x_1, \cdots, x_n\}$ near q, we can express P as

$$P(u) = \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + c$$

for some smooth functions $a_{ij}(x), b_i(x), c(x)$ where (a_{ij}) is uniformly elliptic. The principal symbol of P at q is

$$\sigma_2(P) = \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j,$$

if $g_{ij} = \delta_{ij}$ at q and we denote $\xi = (\xi_1, \cdots, \xi_n)$ using canonical orthonormal basis on $T_q^*\Sigma$.

Remark 4.11. *This thesis only considers smooth second-order elliptic operators to avoid very tedious technical details. The smooth case is already enough for our later application.*

4.3.2 Description using tensors

This subsection shows that second-order elliptic operators can be viewed as a linear combination of several forms acting on the derivative of functions.

We already know if $P \in \mathcal{P}^{(0)}(\Sigma)$, then we can find a smooth function f on Σ such that $P(u) = fu$ for any $u \in C^\infty(\Sigma)$. For $P \in \mathcal{P}^{(1)}(\Sigma)$, we can describe P in the following ways.

Proposition 4.12. *If $P \in \mathcal{P}^{(0)}(\Sigma)$, then there exists a smooth vector field X on Σ and a smooth function f on Σ such that*

$$P(u) = \frac{\partial}{\partial X}(u) + fu, \quad \forall u \in C^\infty(\Sigma).$$

Proof. Note that by definition of $\mathcal{P}^{(0)}(\Sigma)$, we know

$$[P, f] \in \mathcal{P}^{(0)}(\Sigma)$$

for any $f \in C^\infty(\Sigma)$. Hence, we may find a smooth function $Q(f)$ related to f such that

$$[P, f](u) = Q(f)u. \tag{4.1}$$

Now we replace f by fg in equation (4.1) for smooth functions f, g, we have

$$P(fgu) - fP(gu) = Q(fg)u, \quad \forall u \in C^{\infty}(\Sigma). \tag{4.2}$$

On the other hand, we replace f by g in equation (4.1) and multiply f, we have

$$fP(gu) - fgP(u) = fQ(g)u, \quad \forall u \in C^{\infty}(\Sigma) \tag{4.3}$$

At last, we replace u by gu in equation (4.1) and get

$$P(fgu) - fP(gu) = gQ(f)u, \quad \forall u \in C^{\infty}(\Sigma) \tag{4.4}$$

Now from equations (4.2), (4.3), and (4.4), we find

$$Q(fg)u = fQ(g)u + gQ(f)u, \quad \forall u \in C^{\infty}(\Sigma).$$

Hence, as a function, we know the operator Q satisfies $Q(fg) = fQ(g) + gQ(f)$. Hence Q should correspond to a smooth vector field X, and the action of Q is just the derivative of functions with respect to X.

Let us consider the operator $P - Q$. Note that for any $f \in C^{\infty}(\Sigma)$, we have

$$[P-Q, f](u) = [P, f](u) - Q(fu) + fQ(u) = Q(f)u + fQ(u) - Q(fu) = 0, \quad \forall u \in C^{\infty}(\Sigma).$$

This implies $P - Q \in \mathcal{P}^{(0)}(\Sigma)$ and hence $(P-Q)(u) = gu$ for some smooth function $g \in C^{\infty}(\Sigma)$. This means, we can write

$$P = Q + g = \frac{\partial}{\partial X} + g.$$

$\square$

Note that we can also understand the vector field X by one tensor (one form) $\omega := X^{\flat}$ using the metric g. So the action of P can be written as

$$P(u) = \omega(\nabla u) + gu.$$

We have a similar result for $P \in \mathcal{P}^{(2)}(\Sigma)$.

Proposition 4.13. *If $P \in \mathcal{P}^{(2)}(\Sigma)$, then there exists a symmetric two-tensor J, a one-tensor ω, and a smooth function $g \in C^\infty(\Sigma)$ such that*

$$P(u) = J(\nabla^2 u) + \omega(\nabla u) + gu, \quad \forall u \in C^\infty(\Sigma).$$

Proof. By definition of $P \in \mathcal{P}^{(2)}(\Sigma)$, we know for any $f \in C^\infty(\Sigma)$, there is a vector field $X = X_f$ and a function $R(f) \in C^\infty(\Sigma)$ such that

$$[P, f](u) = \frac{\partial}{\partial X_f} u + R(f)u.$$

Using similar calculation as in Proposition 4.12, we have the identity

$$\frac{\partial}{\partial X_{fg}} u + R(fg)u - g\frac{\partial}{\partial X_f}(u) - u\frac{\partial}{\partial X_f} g - gR(f)u - f\frac{\partial}{\partial X_g} u - fR(g)u = 0, \quad \forall u \in C^\infty(\Sigma). \tag{4.5}$$

Hence, we have

$$X_{fg} = fX_g + gX_f, \tag{4.6}$$

$$R(fg) = fR(g) + gR(f) + \frac{\partial}{\partial X_f} g. \tag{4.7}$$

The identity (4.6) implies, the vector X_f at a point $q \in \Sigma$ only depends on the values ∇f at q. Hence, we can think of X as a tensor field of $T^*M \otimes TM$, which maps ∇f to another vector field X_f. We write J as the covariant 2-tensor obtained by $\frac{1}{2}X$ using bundle isomorphisms determined by metric g. That means J can be computed by

$$J(\nabla f \otimes \nabla g) = J(\nabla f, \nabla g) = \frac{1}{2}\frac{\partial}{\partial X_f} g.$$

Note that J is a symmetric 2-tensor in view of identity (4.5). J can also act on $\nabla^2 f$ since $\nabla^2 f$ is a 2-vector.

Let us consider the operator $Q(f) := R(f) - J(\nabla^2 f)$ and using identity (4.7), we have

$$
\begin{aligned}
&Q(fg) = R(fg) - J(\nabla^2(fg)) \\
&= fR(g) + gR(f) + 2J(\nabla f, \nabla g) - gJ(\nabla^2 f) - fJ(\nabla^2 g) - J(\nabla f \otimes \nabla g) - J(\nabla g \otimes \nabla f) \\
&= fQ(g) + gQ(f).
\end{aligned}
$$

Hence, there is a one-form ω such that $Q(f) = \omega(\nabla f)$.

Now let us consider the operator $S(u) := P(u) - J(\nabla^2 u) - \omega(\nabla u)$. It is easy to find

$$[S, f](u) = 0, \quad \forall u \in C^\infty(\Sigma).$$

Hence, there is a smooth function $g \in C^\infty(\Sigma)$ such that $S(u) = gu$. In summary, we get

$$P(u) = J(\nabla^2 u) + \omega(\nabla u) + gu, \quad \forall u \in C^\infty(\Sigma).$$

□

Note that for any $P(u) = J(\nabla^2 u) + \omega(\nabla u) + gu$ that appeared in Proposition 4.13, we can also write P in the divergence form as

$$P(u) = \mathrm{div}(J \cdot \nabla u) - \mathrm{div} J(\nabla u) + \omega(\nabla u) + gu$$

by the properties of divergence for tensors. It is easy to see that P is elliptic if and only if J is a positive defined bilinear form at $T_q\Sigma$ for any $q \in \Sigma$. If P is a (formally) self-adjoint operator, we can find $\omega(\nabla u) = \mathrm{div} J(\nabla u)$. Hence for the second-order self-adjoint elliptic operator P, we can write it as $P(u) = \mathrm{div}(J \cdot \nabla u) + gu$ shortly.

4.4 First and second variation formulas

This section will find that the second-order elliptic operators naturally arise from the first and second variations. This is one of the reasons why we focus on the general second elliptic operators on triple junction hypersurfaces.

Suppose we have an extrinsic triple junction hypersurface $\phi : \Sigma \to N$ for some $(n+1)$-

dimensional Riemannian manifold (N, g_N). Hence, each Σ^i will have the pullback metric $g^i := \left(\phi^i\right)^* (g_N)$.

4.4.1 First variation of Σ

Definition 4.14. We say $\phi_t(\cdot) = \phi(\cdot, t) : \Sigma \times (-\varepsilon, \varepsilon) \to N$ is a *smooth variation* of ϕ, if each ϕ_t^i is a smooth variation of Σ^i up to the boundary and each ϕ_t is an extrinsic triple junction hypersurface in N. We say a smooth variation ϕ_t of ϕ has *compact support* (in the interior of Σ) if $\phi_t = \phi$ near $\partial\Sigma$.

In other words, if a smooth variation ϕ_t has compact support, then the boundary $\partial\Sigma$ will be fixed. Note that we do not fix Γ.

For any smooth variation ϕ_t of ϕ, there is an associated vector field $X : \Sigma \to TN$, such that each X^i is a smooth vector field along Σ^i, vanishes near Γ^i, and $X^i|_\Gamma = X^j|_\Gamma$ for $1 \leq i, j \leq 3$.

We assume Σ has density θ. Then the area of $\phi_t(\Sigma)$ is given by

$$|\phi_t(\Sigma)| = \sum_{i=1}^{3} \int_{\Sigma^i} \theta^i d\Sigma^i_{\phi_t^{i*}(g_N)}.$$

Suppose ϕ_t is a smooth variation for the extrinsic triple junction hypersurface $\phi : \Sigma \to N$ with compact support. Let X be the associated variational vector field. We write the function $f = \langle X, \nu \rangle_{g_N}$, where ν is a unit normal vector field along Σ.

We use the well-known first variation formulas for smooth hypersurfaces and compute $\left.\frac{d}{dt}\right|_{t=0} |\phi_t(\Sigma)|$,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} |\phi_t(\Sigma)| =& \sum_{i=1}^{3} \int_{\Sigma^i} \theta^i \mathrm{div}_{\Sigma^i} X d\Sigma^i_{g^i} \\ =& -\sum_{i=1}^{3} \int_{\Sigma^i} \theta^i H_{\Sigma^i} \left\langle X, \nu^i \right\rangle_{g_N} d\Sigma^i_{g^i} + \sum_{i=1}^{3} \int_{\Gamma} \theta^i \left\langle X, \tau^i \right\rangle_{g_N} d\Gamma_g \end{aligned} \tag{4.8}$$

where H_{Σ^i} is the mean curvature of Σ^i respect to the unit normal vector field ν^i. Now, we can find ϕ is a critical point of the area function if and only if

$$H_{\Sigma^i} \equiv 0 \text{ on } \Sigma^i \quad \text{and} \quad \sum_{i=1}^{3} \theta^i \tau^i \equiv 0 \text{ on } \Gamma. \tag{4.9}$$

Definition 4.15. We say an extrinsic triple junction hypersurface $\phi : \Sigma \to N$ with density θ is *minimal* if the conditions (4.9) holds for Σ.

Remark 4.16. *Note that* $\sum_{i=1}^{3} \theta^i \tau^i = 0$ *can hold only when* $\theta^i + \theta^i \geq \theta^k$ *for any permutation* (ijk) *of* (123)*. Note that it should be a strict inequality if we recall the non-degenerate property in definition of extrinsic triple junction hypersurfaces.*

If we suppose the ambient manifold $(N, g_N) = (\mathbb{R}^{n+1}, \delta_{ij})$ is the standard Euclidean space, we see that minimal immersions are equivalent to the following properties of coordinate functions. Note that $H_{\Sigma^i} \equiv 0$ is equivalent to the equation $\Delta_{\Sigma^i} x_j \equiv 0$ where $x_j := x_j \circ \phi$ is the coordinate function for Σ. On the other hand, the vector field τ can also be viewed as the derivative of coordinate function (e.g., $\tau = \frac{\partial x}{\partial \tau}$). Hence, we know the coordinate x_j should solve the following problem,

$$\begin{cases} \Delta_\Sigma x_j = 0, & \text{on } \Sigma \\ x_j \in C^{(1)}_{E_1, E_\theta}, \end{cases} \tag{4.10}$$

Moreover, it is easy to note the following result.

Proposition 4.17. *Suppose* Σ *is an extrinsic triple junction hypersurface in* $\mathbb{R}^{n+1}$*. Then* Σ *is minimal with respect to the density* θ *if and only if every coordinate function* x_j *satisfies Problem (4.10) for* $1 \leq j \leq n+1$*.*

Proposition 4.17 shows, it is worth to study the operator Δ_Σ. In later applications, we will use similar results like an immersion of Σ to a unit sphere is a minimal immersion if and only if the coordinate functions solve particular elliptic partial differential equations.

4.4.2 Second variation of Σ

We assume ϕ_t is a smooth variation of $\phi : \Sigma \to N$ with compact support and X is the associated vector field. We write $f = \langle X, \nu \rangle_{g_N}$. Then by the similar calculation that appeared in [Wan22], we can get the following formula.

Proposition 4.18. *Suppose $\phi : \Sigma \to N$ is a minimal extrinsic triple junction hypersurface. The for any smooth variation ϕ_t of ϕ, we can compute the second derivative of its area by*

$$\begin{aligned} \frac{d^2}{dt^2}\bigg|_{t=0} |\phi_t(\Sigma)| = & \sum_{i=1}^{3} \int_{\Sigma^i} \left[\left| \nabla_{\Sigma^i} f^i \right|^2 - |A_{\Sigma^i}|^2 (f^i)^2 - \mathrm{Ric}^N(\nu^i)(f^i)^2 \right] \theta^i d\Sigma^i \\ & - \sum_{i=1}^{3} \int_{\Gamma} (f^i)^2 \left\langle \boldsymbol{H}_\Gamma, \tau^i \right\rangle_{g_N} \theta^i d\Gamma. \end{aligned} \tag{4.11}$$

We can shorten the formula (4.11) as

$$\frac{d^2}{dt^2}\bigg|_{t=0} |\phi_t(\Sigma)| = \int_\Sigma \theta \left[|\nabla_\Sigma f|^2 - |A_\Sigma|^2 f^2 - \mathrm{Ric}^N(\nu) f^2 \right] d\Sigma - \int_\Gamma \theta f^2 \langle \boldsymbol{H}_\Gamma, \tau \rangle_{g_N} d\Gamma$$

using the notations in Section 4.2.

Proof. We will use the methods in [RS97, Appendix] (see also [Wan22, Theorem 4]).

Let $(\cdot)'$ be $\frac{D}{dt}|_{t=0}(\cdot)$. Here, we use D to denote the covariant derivative on N. Take derivative with respect to t in the first variation formula (4.8), and we get

$$\frac{d^2}{dt^2}\bigg|_{t=0} |\phi_t(\Sigma)| = -\int_\Sigma \theta H'_\Sigma f d\Sigma + \int_\Gamma \theta \left\langle X, \tau' \right\rangle_{g_N} d\Gamma. \tag{4.12}$$

Note that here, we have used minimal conditions $H_\Sigma \equiv 0$ and $\sum_{i=1}^3 \theta^i \tau^i = 0$ along Γ. Using the well-known formula (cf. [Ros93]), we have $H'_\Sigma = \Delta_\Sigma f + |A_\Sigma|^2 f + \mathrm{Ric}^N(\nu) f$. This will contribute to the interior term in the formula (4.11).

Now, let us compute $\tau' = ((\tau^1)', (\tau^2)', (\tau^3)')$.

Let $S_\Sigma = (S_{\Sigma^i})_{i=1}^3$ denote the shape operator of Σ with respect to the normal ν (i.e., $\langle S_\Sigma(X), Y \rangle_{g_N} = A_\Sigma(X, Y)$ for $X, Y \in \mathfrak{X}(\Sigma)$). Let X_Σ^T, X_Γ^T denote the tangential component of vector fields X when immersing Σ and Γ, respectively, into N.

Using Lemma 4.1 in [RS97], we have

$$\nu' = -\nabla_\Sigma(f) - S_\Sigma(X_\Sigma^T).$$

Let $\{e_j\}_{j=1}^{n-1}$ be an orthonormal frame of $T_p\Gamma$ for some $p \in \Gamma$. We write $e_j(t) = d\phi_t(e_i)$ and decompose $X = f\nu + g\tau + X_\Gamma^T$ where $g = \langle X, \tau \rangle$. Then we find

$$\left\langle e_j', \tau \right\rangle_{g_N} = \left\langle D_{e_j} X, \tau \right\rangle_{g_N} = -f A_\Sigma(e_j, \tau) + \frac{\partial g}{\partial e_j} - A_\Gamma(X_\Gamma^T, e_j),$$

for $1 \le j \le n-1$. Recall that $A_\Gamma = (A_\Gamma^1, A_\Gamma^2, A_\Gamma^3)$.

Hence, we can compute $\langle X, \tau' \rangle_{g_N}$ to get

$$\begin{aligned}
\langle X, \tau' \rangle_{g_N} &= \langle \tau', \nu \rangle_{g_N} f + \langle \tau', \tau \rangle_{g_N} g + \sum_{j=1}^{n-1} \langle \tau', e_j \rangle_{g_N} \left\langle X_\Gamma^T, e_j \right\rangle_{g_N} \\
&= - \langle \tau, \nu' \rangle_{g_N} f + \sum_{j=1}^{n-1} \left\langle \tau, e_j' \right\rangle_{g_N} \left\langle X_\Gamma^T, e_j \right\rangle_{g_N} \\
&= f \frac{\partial f}{\partial \tau} + f A_\Sigma(\tau, X_\Sigma^T) + f A_\Sigma(X_\Gamma^T, \tau) - \frac{\partial g}{\partial X_\Gamma^T} + A_\Gamma(X_\Gamma^T, X_\Gamma^T) \\
&= f \frac{\partial f}{\partial \tau} + fg A_\Sigma(\tau, \tau) + 2f A_\Sigma(X_\Gamma^T, \tau) - \frac{\partial g}{\partial X_\Gamma^T} + A_\Gamma(X_\Gamma^T, X_\Gamma^T).
\end{aligned} \tag{4.13}$$

Note that the last three terms in identity (4.13) are sections of E_θ. Indeed, we can find $g \in \Gamma(E_\theta)$ and $\sum_{i=1}^3 \theta^i A_\Gamma^i = 0$ by Proposition 3.26. For the term $f A_\Sigma(X_\Gamma^T, \tau)$, we have

$$\begin{aligned}
&\sum_{i=1}^3 \theta^i f^i A_\Sigma^i(X_\Gamma^T, \tau^i) = \sum_{i=1}^3 \theta^i \left\langle D_{X_\Gamma^T} \tau^i, f^i \nu^i \right\rangle_{g_N} \\
= &\sum_{i=1}^3 \left\langle D_{X_\Gamma^T}(\theta^i \tau^i), X \right\rangle_{g_N} - \sum_{i=1}^3 \left\langle D_{X_\Gamma^T}(\theta^i \tau^i), X_\Gamma^T \right\rangle_{g_N} - \sum_{i=1}^3 \theta^i g^i \left\langle D_{X_\Gamma^T} \tau^i, \tau^i \right\rangle_{g_N} = 0,
\end{aligned}$$

using $\sum_{i=1}^3 \theta^i \tau^i \equiv 0$ and $\langle \tau^i, \tau^i \rangle_{g_N} \equiv 1$ along Γ.

For the term $fg A_\Sigma(\tau, \tau)$, using minimal conditions, we have

$$\begin{aligned}
&\sum_{i=1}^3 \theta^i f^i g^i A_{\Sigma^i}(\tau^i, \tau^i) = \sum_{i=1}^3 \sum_{j=1}^{n-1} -\theta^i f^i g^i A_{\Sigma^i}(e_j, e_j) = -\sum_{i=1}^3 \theta^i g^i \left\langle \boldsymbol{H}_\Gamma, f^i \nu^i \right\rangle_{g_N} \\
= \; &-\sum_{i=1}^3 \theta^i g^i \left[\langle \boldsymbol{H}_\Gamma, X \rangle_{g_N} - g^i \left\langle \boldsymbol{H}_\Gamma, \tau^i \right\rangle_{g_N} - \left\langle \boldsymbol{H}_\Gamma, X_\Gamma^T \right\rangle_{g_N} \right] = \sum_{i=1}^3 \theta^i (g^i)^2 \left\langle \boldsymbol{H}_\Gamma, \tau^i \right\rangle_{g_N}.
\end{aligned}$$

Note that $g^2 = |X|_{g_N}^2 - f^2 - \left|X_\Gamma^T\right|_{g_N}^2$ where $|\cdot|_{g_N}^2 := \langle \cdot, \cdot \rangle_{g_N}$. Using the fact $\sum_{i=1}^3 \theta^i \tau^i = 0$

along Γ, we have

$$\sum_{i=1}^{3}\theta^i f^i g^i A_{\Sigma^i}(\tau^i,\tau^i) = -\sum_{i=1}^{3}\theta^i (f^i)^2 \left\langle \boldsymbol{H}_\Gamma, \tau^i \right\rangle_{g_N}.$$

Combining with the identity (4.13) and (4.12), we have

$$\frac{d^2}{dt^2}\bigg|_{t=0} |\phi_t(\Sigma)| = \int_\Sigma \theta f \left(-\Delta_\Sigma f - |A_\Sigma|^2 f - \mathrm{Ric}^N(\nu) f\right) d\Sigma + \int_\Gamma \theta f \left(\frac{\partial f}{\partial \tau} - f \left\langle \boldsymbol{H}_\Gamma, \tau \right\rangle_{g_N}\right) d\Gamma. \tag{4.14}$$

After integration by parts, we can get the formula (4.11). □

In view of formula (4.14), we know that the second-order elliptic operator $J := \Delta_\Sigma + |A_\Sigma|^2 + \mathrm{Ric}^N(\nu)$ (the usual Jacobi operator) plays an essential role in the second variation formula. In particular, there is an additional boundary term on Γ, which is associated with the function $\langle \boldsymbol{H}_\Gamma, \tau \rangle_{g_N}$. We view this function as a part of elliptic operators. Please see the next chapter for details.

Chapter 5

Elliptic partial differential equations on triple junction hypersurfaces

In this chapter, we will study second-order elliptic partial differential equations on the triple junction hypersurface Σ. We will only focus on self-adjoint operators since their results can be easily adapted to the general case. We will make the same assumptions for Σ used in Chapter 4.

5.1 Definitions of problems

In this chapter, we will be interested in the following problem,

$$\begin{cases} -Lu = f, & \text{in } \Sigma, \\ u = 0, & \text{on } \partial\Sigma, \\ u|_\Gamma \in \Gamma(E), & \\ (J(\nabla u, \tau) + uh)|_\Gamma \in \Gamma(E^\perp) & \end{cases} \tag{5.1}$$

for smooth subbundle $E = E_1$ or $E = F_\theta$. Here, the function f, defined on Σ, is given, and h, a section of $\Gamma \times \mathbb{R}^3$, is given. u is the unknown function. We suppose L is the self-adjoint second-order elliptic operator on Σ and write $L(u)$ as $L(u) = \operatorname{div}(J \cdot \nabla u) - cu$ for some symmetric two-tensor J and smooth function $c \in C^\infty(\Sigma)$. Note that we understand J as $J = (J^1, J^2, J^3)$.

Usually, the function h will be related to the geometric properties of Σ, and we will always assume that it is a smooth section of $\Gamma \times \mathbb{R}^3$. Hence, we can view h as a particular coefficient of L defined on Γ.

We will use (L, h) to denote the symmetric elliptic operators on Σ that may have the non-zero coefficient h on Γ.

5.1.1 Weak solutions

Assume we have a smooth solution to Problem (5.1). For any $v \in C_E^{(0)}(\Sigma)$ that vanishes on $\partial\Sigma$, we can multiply $v\theta$ to $Lu = 0$ and integrate on Σ to get the following identity,

$$\int_\Sigma \left[-v\mathrm{div}(J \cdot \nabla u) + cuv\right] \theta d\Sigma = \int_\Sigma fv\theta d\Sigma.$$

Then we can apply integration by parts to get

$$\int_\Sigma \left[J(\nabla u, \nabla v) + cuv\right] \theta d\Sigma - \int_\Gamma J(\nabla u, \tau) v\theta d\Gamma = \int_\Sigma fv\theta d\Sigma. \tag{5.2}$$

From the fourth condition in Problem (5.1), we know

$$\int_\Gamma (J(\nabla u, \tau) + uh) v\theta d\Gamma = 0.$$

Put the above identity into equation (5.2), we get the following identity,

$$\int_\Sigma \left[J(\nabla u, \nabla v) + cuv\right] \theta d\Sigma + \int_\Gamma uvh\theta d\Gamma = \int_\Sigma fv\theta d\Sigma.$$

Now we can define the bilinear form $B[\cdot, \cdot]$ as

$$B[u, v] := \int_\Sigma \left[J(\nabla u, \nabla v) + cuv\right] \theta d\Sigma + \int_\Gamma uvh d\Gamma. \tag{5.3}$$

Note that although B is defined only for smooth functions initially, we can extend the definition of J to ensure B is defined on $H^1(\Sigma)$ since smooth functions are dense in $H^1(\Sigma)$.

Remark 5.1. *We can compare the bilinear form B in equation (5.3) with the second variation formula*

(4.11). Note that the function h will corresponding to the term $\langle \boldsymbol{H}_\Gamma, \tau \rangle_{g_N}$ in equation (4.11). This is the reason why we have an extra h coming into the definition of Problem (5.1). In our case, the function h is determined by the geometric properties of Σ. Hence it is reasonable to assume h is given and smooth on Γ.

Definition 5.2. We say that $u \in H^1_{0,E}(\Sigma)$ is a *weak solution* to Problem (5.1) if for any $v \in H^1_{0,E}(\Sigma)$, we have

$$B[u, v] = (f, v)_{L^2_\theta(\Sigma)},$$

where $(\cdot, \cdot)_{L^2_\theta(\Sigma)}$ is the L^2 inner product on Σ with respect to the density θ.

5.2 Existence of weak solutions

First of all, let us show a preliminary energy estimate for later application.

Proposition 5.3. *There exist some constants $\alpha, \beta, \gamma > 0$ such that*

$$\alpha \|u\|^2_{H^1_\theta(\Sigma)} \le B[u, u] + \gamma \|u\|^2_{L^2_\theta(\Sigma)} \le \beta \|u\|^2_{H^1_\theta(\Sigma)}$$

for all $u \in H^1_0(\Sigma)$. Here, we write $H^k_0(\Sigma) := W^{k,2}_0(\Sigma)$. The constants only depend on Σ, the operator L and the L^∞ norm of h.

Before the proof of this proposition, let us prove a trace theorem which is required in this proof.

Lemma 5.4. *For any fixed $C > 0$, there exists a constant Λ large enough such that,*

$$\|\nabla u\|^2_{L^2_\theta(\Sigma)} + \Lambda \|u\|^2_{L^2_\theta(\Sigma)} \ge C \|u\|^2_{L^2_\theta(\Gamma)}$$

for any $u \in H^1_0(\Sigma)$.

Recall that $\|u\|^2_{L^2_\theta(\Gamma)} := \int_\Gamma u^2 \theta d\Gamma$.

Proof of Lemma 5.4. Note that by trace theorem for $u \in W^{1,1}_0(\Sigma)$, we can find a constant c_1 such that

$$\int_\Gamma |u|\, \theta d\Gamma \le c_1 \left[\int_\Sigma |\nabla u|\, \theta d\Sigma + \int_\Sigma |u|\, \theta d\Sigma \right].$$

Let us replace u by u^2, then by Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_\Gamma u^2\theta d\Gamma &\le c_1\left[\int_\Sigma 2\,|u|\,|\nabla u|\,\theta d\Sigma + \int_\Sigma u^2\theta d\Sigma\right]\\ &\le c_1\varepsilon\int_\Sigma |\nabla u|^2\,\theta d\Sigma + \left(\frac{c_1}{\varepsilon}+c_1\right)\int_\Sigma u^2\theta d\Sigma.\end{aligned}$$

Hence, we can choose ε small enough to ensure $\frac{c_1\varepsilon}{2} < \frac{1}{C}$, so we have

$$C\|u\|^2_{L^2_\theta(\Sigma)} \le \|\nabla u\|^2_{L^2_\theta(\Sigma)} + \Lambda\|u\|^2_{L^2_\theta(\Sigma)}$$

for some Λ sufficiently large. □

Proof of Proposition 5.3. Let us find some α, γ such that the inequality

$$\alpha\|u\|^2_{H^1_\theta(\Sigma)} \le B[u,u] + \gamma\|u\|^2_{L^2_\theta(\Sigma)}$$

holds.

Note that since L is elliptic, we can find $\lambda > 0$ such that

$$\frac{1}{\lambda}\,|\xi|^2 \le J(\xi,\xi) \le \lambda\,|\xi|^2\,. \tag{5.4}$$

Since the L^∞ norm of h is bounded, we know there exists a constant c_1 such that

$$\left|\int_\Sigma hu^2\theta d\Gamma\right| \le c_1\|u\|^2_{L^2_\theta(\Gamma)}. \tag{5.5}$$

Using Lemma 5.4, we can find Λ such that

$$c_1\|u\|^2_{L^2_\theta(\Gamma)} \le \frac{1}{2\lambda}\|\nabla u\|^2_{L^2_\theta(\Sigma)} + \Lambda\|u\|^2_{L^2_\theta(\Sigma)}. \tag{5.6}$$

Hence, combining the results of (5.4), (5.5), (5.6) and choose Λ' such that $\Lambda' > |c|$, then we have

$$B[u,u] + \left(\frac{1}{2\lambda} + \Lambda + \Lambda'\right)\|u\|^2_{L^2_\theta(\Sigma)} \ge \frac{1}{2\lambda}\|\nabla u\|^2_{L^2_\theta(\Sigma)} + \frac{1}{2\lambda}\|u\|^2_{L^2_\theta(\Sigma)} = \frac{1}{2\lambda}\|u\|^2_{H^1_\theta(\Sigma)}.$$

Hence we can choose $\alpha = \frac{1}{2\lambda}, \gamma = \frac{1}{2\lambda} + \Lambda + \Lambda'$ to make the first inequality holds.

For the second inequality, it is easy to see by inequality (5.6) as

$$B[u,u]+\gamma\|u\|^2_{L^2_\theta(\Sigma)} \le \lambda\|\nabla u\|^2_{L^2_\theta(\Sigma)} + \frac{1}{2\lambda}\|\nabla u\|^2_{L^2_\theta(\Sigma)} + (\gamma+\Lambda+\Lambda')\|u\|_{L^2_\theta(\Sigma)} \le \beta\|u\|^2_{H^1_\theta(\Sigma)}$$

for β sufficiently large. □

Using the Fredholm alternative, we can get the existence theorem for the weak solutions to Problem (5.1).

Let $N \subset H^1_{0,E}(\Sigma)$ be the spaces of functions u which solve Problem (5.1) with $f \equiv 0$ (homogeneous problem) weakly.

Theorem 5.5. *Problem (5.1) has a weak solution if and only if $(f,v)_{L^2_\theta(\Sigma)} = 0$ for any $v \in N$. In particular, the solution u is unique in the sense of $(u,v)_{L^2_\theta(\Sigma)} = 0$ for any $v \in N$.*

Remark 5.6. *If N only contains zero function, then Theorem 5.5 says for any $f \in L^2(\Sigma)$, there exists a unique weak solution u to Problem (5.1).*

Proof of Theorem 5.5. Let us define the bilinear form

$$B_\gamma[u,v] := B[u,v] + \gamma(u,v)_{L^2_\theta(\Sigma)}$$

where we choose γ as in Proposition 5.3. Note that by Proposition 5.3, we know B_γ indeed defines a norm on Σ and this norm is equivalent to the usual norm $\|\cdot\|_{H^1_\theta(\Sigma)}$.

Note that the function f determines a bounded linear functional on $H^1_{0,E}(\Sigma)$ as

$$T_f : v \to (f,v)_{L^2_\theta(\Sigma)}.$$

Hence, by Riesz Representation Theorem, there exists a unique element $u \in H^1_{0,E}(\Sigma)$ such that

$$B_\gamma[u,v] = (f,v)_{L^2_\theta(\Sigma)}, \quad \forall v \in H^1_{0,E}(\Sigma).$$

Hence, we get a map from f to u. Let us write this map as $u = P(f)$. It is easy to find P is indeed a linear operator. Note that $u \in H^1_{0,E}(\Sigma)$ is a weak to Problem (5.1) if and only if $B_\gamma[u,v] = (\gamma u + f, v)_{L^2_\theta(\Sigma)}$ for any $v \in H^1_{0,E}(\Sigma)$. This is equivalent to the following identity

$$u - \gamma P(u) = P(f). \tag{5.7}$$

Now let us show P is a bounded operator. Indeed, by Proposition 5.3, we have

$$\alpha\|P(f)\|^2_{H^1_\theta(\Sigma)} \leq B_\gamma[P(f), P(f)] = (P(f), f)_{L^2_\theta(\Sigma)} \leq \|P(f)\|_{H^1_\theta(\Sigma)}\|f\|_{L^2_\theta(\Sigma)}.$$

This implies $\alpha\|P(f)\|_{H^1_\theta(\Sigma)} \leq \|f\|_{L^2_\theta(\Sigma)}$. Hence P is a bounded linear operator $P : L^2(\Sigma) \to H^1_{0,E}(\Sigma) \subset L^2(\Sigma)$. Since $H^1_{0,E}(\Sigma)$ is compactly embedded in $L^2(\Sigma)$, we actually know P is a bounded, linear, compact operator.

So we may apply the Fredholm alternative for the operator γP.

Let us write $\overline{N} := \left\{u \in H^1_{0,E}(\Sigma) : u - \gamma P(u) = 0\right\}$. Then from Fredholm alternative, we know the equation (5.7) has a solution $u \in H^1_{0,E}(\Sigma)$ if and only if $P(f) \perp \overline{N}$ under the inner product $(\cdot, \cdot)_{L^2_\theta(\Sigma)}$. The solution is unique if we require $u \perp \overline{N}$.

Note that

$$\begin{aligned}
u \in \overline{N} &\iff B_\gamma[u, v] - \gamma B_\gamma[P(u), v] = 0, \quad \forall v \in H^1_{0,E}(\Sigma) \\
&\iff B[u, v] + \gamma(u, v)_{L^2_\theta(\Sigma)} - \gamma(u, v)_{L^2_\theta(\Sigma)}, \quad \forall v \in H^1_{0,E}(\Sigma), \\
&\iff B[u, v] = 0, \quad \forall v \in H^1_{0,E}(\Sigma).
\end{aligned}$$

Hence, we know $\overline{N}$ is the same with N. On the other hand, for any $u \in \overline{N}$, we note

$$B_\gamma[u, v] = 0 \iff B_\gamma[\gamma P(u), v] = 0 \iff \gamma(u, v)_{L^2_\theta(\Sigma)} = 0, \quad \forall v \in H^1_\theta(\Sigma)$$
$$(P(f), u)_{L^2_\theta(\Sigma)} = 0 \iff B_\gamma[P(f), u] = 0 \iff (f, u)_{L^2_\theta(\Sigma)} = 0, \quad \forall f \in L^2(\Sigma).$$

Hence, $P(f) \perp \overline{N}$ is equivalent to the $(f, v)_{L^2_\theta(\Sigma)} = 0$ for all $v \in N$. Under this condition, the solution u will exist and it is unique if we require $(u, v)_{L^2_\theta(\Sigma)} = 0$ for any $v \in N$. $\square$

5.3 Regularity

In this section, we will show that a weak solution to Problem (5.1) can be improved to a smooth solution provided f is sufficiently regular. In particular, this solution should be a classical solution to Problem (5.1).

Theorem 5.7. *Suppose $f \in L^2(\Sigma)$ and $u \in H^1_{0,E}(\Sigma)$ is a weak solution to Problem (5.1). Then*

$$u \in H^2(\Sigma),$$

with estimate

$$\|u\|_{H^2_\theta(\Sigma)} \leq C\left(\|f\|_{L^2_\theta(U)} + \|u\|_{H^1_\theta(\Sigma)}\right). \tag{5.8}$$

Proof. The proof is quite standard, and it is similar to the proof of regularity for usual elliptic PDEs.

Note that by the regularity of usual second-order elliptic PDEs, we know u is H^2 away from the triple junction Γ. We only need to show the regularity near the junction.

Since the regularity is purely a local result, we can choose a coordinate chart described in Proposition 3.25 for any $p \in \Gamma$ such that Σ can be written as the union of three half balls $B_{\delta+} = B_{\delta+}(0) := (B^1_{\delta+}(0), B^2_{\delta+}(0), B^3_{\delta+}(0))$ near p for some δ small enough. Here, $B^i_{\delta+}(0) := \{x = (x_1, \cdots, x_n) \in B_\delta(0) : x_1 \geq 0, |x| \leq \delta\}$.

Note that we can view $B_{\delta+}$ as a triple junction hypersurface with corner in the sense that we choose $\Gamma^i = \partial B^i_{\delta+} \cap \{|x| = \delta\}$ and $\mathring{\Gamma}^i = \partial B^i_{\delta+} \cap \{x_1 = 0\}$. The outer unit normal vector field τ is given by $\tau^i = -\frac{\partial}{\partial x_1}$.

So we can write the operator L in the local coordinate as

$$L(u) = \left(a_{ij} u_{x_i}\right)_{x_j} - cu$$

for some smooth function a_{ij}, c defined on $B_{\delta+}$. Here, we have used Einstein summation. Note that since u is a weak solution, we can find

$$\int_{B_{\delta+}} \left[a_{ij} u_{x_i} v_{x_j} + cuv\right] \theta dx + \int_{\Gamma \cap B_{\delta+}} huv\theta dy = \int_{B_{\delta+}} fv\theta dx, \quad \forall v \in H^1_{0,E}(B_{\delta+}). \tag{5.9}$$

where we use y to denote the coordinate of points in $\mathbb{R}^{n-1}$. Here, we can view the space $H^1_{0,E}(B_{\delta+})$ as the function $v \in H^1_{0,E}(\Sigma)$ which is supported in $B_{\delta+}$ and $v = 0$ on $\partial B_{\delta+}$ in the trace sense.

We write $U = B_{\delta+}$ and $V = B_{\frac{\delta}{2}+}$ for short. Let us choose a smooth cutoff function ζ

such that

$$\zeta = 1 \text{ in } V, \quad 0 \le \zeta \le 1, \quad \zeta \text{ is supported in } B_{\frac{3\delta}{4}+}.$$

Recall that here, we think ζ as $\zeta = (\zeta^1, \zeta^2, \zeta^3)$, each ζ^i is defined on $B^i_{\delta+}(0)$. So the function $\zeta \in C^{(0)}_{E_1}(\Sigma)$.

For any $t \neq 0$, we define the differential quotient D^t_k as

$$D^t_k u = \frac{u(x+te_k) - u(x)}{t}$$

where $\{e_1, \cdots, e_n\}$ is the standard orthonormal basis in Euclidean space. Note that if $u \in H^1_{0,E}(\Sigma)$, then $u(x+te_k)|_\Gamma$ is a section of E near p for t small enough and $k \neq 1$. Note that here, we have used the fact $E = E_1$ or $E = F_\theta$. This implies, we can choose $|t| > 0$ small enough, $k \in \{2, \cdots, n\}$ to ensure $v := -D^{-t}_k(\zeta^2 D^t_k u) \in H^1_{0,E}(B_{\delta+})$. (Note that we have used results like Proposition 3.12 to ensure $v \in H^1_{0,E}(B_{\delta+})$.) This is the only part we need the assumption $E = E_1$ or $E = F_\theta$.

We put v into the identity (5.9), and get

$$\int_U a_{ij} u_{x_i} v_{x_j} \theta dx + \int_{\Gamma \cap U} huv\theta dy = \int_U \tilde{f} v \theta dx \tag{5.10}$$

for $\tilde{f} = f - cu$. We use the following notations

$$\begin{aligned} \mathrm{I} &:= \int_U a_{ij} u_{x_i} v_{x_j} \theta dx, \\ \mathrm{II} &:= \int_{\Gamma \cap U} huv\theta dy, \\ \mathrm{III} &:= \int_U \tilde{f} v \theta dx. \end{aligned}$$

Now, by an elementary computation, we have

$$\begin{aligned} \mathrm{I} = & \int_U D^t_k (a_{ij} u_{x_i}) (\zeta^2 D^t_k u)_{x_j} \theta dx \\ = & \int_U a^t_{ij} D^t_k u_{x_i} \zeta^2 D^t_k u_{x_j} \theta dx \\ & + \int_U \left[2a^t_{ij} D^t_k u_{x_i} \zeta \zeta_{x_j} D^t_k u + D^t_k a_{ij} u_{x_i} D^t_k u_{x_j} \zeta^2 + 2 D^t_k a_{ij} u_{x_i} D^t_k u \zeta \zeta_{x_j} \right] \theta dx \\ =: & \mathrm{I}_1 + \mathrm{I}_2, \end{aligned} \tag{5.11}$$

where $a_{ij}^t := a_{ij}(x + te_k)$.

Note that we have

$$\mathrm{I}_1 \geq \frac{1}{\lambda} \int_U \zeta^2 \left| D_k^t Du \right|^2 \theta dx \tag{5.12}$$

by the elliptic condition for some λ large. By Cauchy-Schwarz inequality, we can estimate the term I_2 by

$$\begin{aligned}
\mathrm{I}_2 &\leq \varepsilon \int_U \zeta^2 \left| D_k^t Du \right|^2 dx + \frac{C}{\varepsilon} \int_U \zeta \left[\left| D_k^t u \right|^2 + |Du|^2 \right] \theta dx \\
&\leq \varepsilon \int_U \zeta^2 \left| D_k^t Du \right|^2 dx + C(\varepsilon) \int_U |Du|^2 \theta dx.
\end{aligned} \tag{5.13}$$

Note that we have used that the differential quotient can be controlled by L^2 norm of Du. Hence we get

$$\mathrm{I} \geq \frac{1}{2\lambda} \int_U \zeta^2 \left| D_k^t Du \right|^2 \theta dx - C \int_U |Du|^2 \theta dx, \tag{5.14}$$

from inequality (5.12), (5.13) by choosing $\varepsilon = \frac{1}{2\lambda}$.

For the estimation on II, we have

$$\begin{aligned}
|\mathrm{II}| &\leq \int_{\Gamma \cap U} \left[|h^t| \left| D_k^t u \right|^2 \zeta^2 + \left| D_k^t hu D_k^t u \right| \right] \theta dy \\
&\leq C \int_{\Gamma \cap U} \left[\zeta^2 \left| D_k^t u \right|^2 + u^2 \right] \theta dy \\
&\leq \frac{1}{8\lambda} \int_U \left| D(\zeta D_k^t u) \right|^2 \theta dx + C \int_U \zeta^2 \left| D_k^t u \right|^2 \theta dx + C \int_U |Du|^2 \theta dx + C \int_U u^2 \theta dx \\
&\leq \frac{1}{4\lambda} \int_U \zeta^2 \left| D_k^t Du \right|^2 \theta dx + C \int_U |Du|^2 \theta dx
\end{aligned} \tag{5.15}$$

by Cauchy-Schwarz inequality. Here we have used Lemma 5.4.

Similarly, we can estimate the term III by

$$\begin{aligned}
|\mathrm{III}| &\leq C \int_U (|f| + |u|) \, |v| \, \theta dx \\
&\leq \varepsilon \int_U \left| D_k^t (\zeta^2 D_k^t u) \right|^2 \theta dx + \frac{C}{\varepsilon} \int_U (f^2 + u^2) \theta dx \\
&\leq C\varepsilon \int_U \left| D(\zeta^2 D_k^t u) \right|^2 \theta dx + \frac{C}{\varepsilon} \int_U (f^2 + u^2) \theta dx \\
&\leq C\varepsilon \int_U \zeta^2 \left| D_k^t Du \right|^2 \theta dx + C(\varepsilon) \int_U (f^2 + u^2 + |Du|^2) dx.
\end{aligned} \tag{5.16}$$

Now we can choose $\varepsilon = \frac{1}{8\lambda C}$ in inequality (5.16) and combining the inequality (5.14) and

(5.15) to get

$$\int_V \left|D_k^t Du\right|^2 \theta dx \leq C \int_U (f^2 + u^2 + |Du|^2)\theta dx$$

for $k = 2, \cdots, n$ and sufficiently small $|t| \neq 0$. Hence, from the properties of differential quotient, we know $u_{x_i} \in H^1(V)$ with estimate

$$\|u_{x_i x_j}\|_{L^2_\theta(V)} \leq C \left(\|f\|_{L^2_\theta(U)} + \|u\|_{H^1_\theta(\Sigma)} \right). \tag{5.17}$$

for $1 \leq i, j \leq n$ except the case $i = j = 1$.

Now we can use the equation to show the estimate (5.17) holds for $i = j = 1$.

Recall that we already know u is in $H^2_{\text{loc}}(\Sigma)$ away from triple junction Γ. Hence, u will satisfy the equation

$$(a_{ij} u_{x_i})_{x_j} + cuv = f, \quad \text{a.e. in } U.$$

Note that the ellipticity of L implies $a_{11} \geq \frac{1}{\lambda}$. Hence, we have

$$|u_{x_1 x_1}| \leq C \sum_{1 \leq i,j \leq n, i+j > 1} \left|u_{x_i x_j}\right| + |Du| + |u| + |f|.$$

Hence, we get $\|u_{x_1 x_1}\|^2_{L^2_\theta(V)} \leq C \left(\|f\|_{L^2_\theta(U)} + \|u\|_{H^1_\theta(\Sigma)} \right)$.

In summary, we get $u \in H^2(V)$ and hence $u \in H^2(\Sigma)$ by the partition of unity with estimate

$$\|u\|_{H^2_\theta(\Sigma)} \leq C \left(\|f\|_{L^2_\theta(U)} + \|u\|_{H^1_\theta(\Sigma)} \right).$$

□

Remark 5.8. *Recall that we used $E = E_1$ or F_θ in the proof of Theorem 5.7. Indeed, we know it is true for other kinds of E such as $E = E_\theta$. But for general smooth E, we conjecture that Theorem 5.7 is valid, too. The tricky part is, we do not know if $D_k^t u|_\Gamma \in \Gamma(E)$ for $u \in H^1_{0,E}(\Sigma)$. Hence, we need to construct a new differential quotient of u such that it maps $H^1_{0,E}(\Sigma)$ to $H^1_{0,E}(\Sigma)$ and shares similar properties with the usual differential quotient.*

Since we know $u \in H^2(\Sigma)$ if u is a weak solution to Problem (5.1) for $f \in L^2(\Sigma)$, we know ∇u is well-defined on Γ in the trace sense. Hence, we can get the following result.

Proposition 5.9. *If u is a weak solution to Problem (5.1) with $f \in L^2(\Sigma)$, then $(J(\nabla u, \tau) + uh)|_\Gamma$ is in $E^\perp$ for a.e. $p \in \Gamma$.*

Note that here, a.e. means almost everywhere with respect to the boundary measure on Γ.

Proof of Proposition 5.9. Note that this property is a local property. Hence, we can use the coordinate chart that appeared in the proof of Theorem 5.7. This means, we can assume the identity (5.9) holds for any $v \in H^1_{0,E}(B_{\delta+})$. We fix such v and apply integration by parts since $u \in H^2(\Sigma)$ to get

$$\int_{B_{\delta+}} \left[-(a_{ij}u_{x_i})_{x_j} + cu\right] v\theta dx + \int_{\Gamma\cap B_{\delta+}} (-a_{i1}u_{x_i} + uh)\, v\theta dy = \int_{B_{\delta+}} fv\theta dx.$$

Note that we already know $-(a_{ij}u_{x_i})_{x_j} + cu = f$ a.e. in Σ. Hence, we get

$$\int_{\Gamma\cap B_{\delta+}} (-a_{i1}u_{x_i} + uh)v\theta dy = 0$$

for any $v \in H^1_{0,E}(B_{\delta+})$. This means $(-a_{i1}u_{x_i} + uh)\perp E$ a.e. near p on Γ with respect to the density θ. Note that $e_1 = -\tau$ under this coordinate chart and we can write $-a_{i1}u_{x_i} = J(\nabla u, \tau)$, we get

$$J(\nabla u, \tau) + hu\perp E, \quad \text{a.e. on } \Gamma.$$

$\square$

Remark 5.10. *From Proposition 5.9, we know the weak solution u should solve Problem (5.1) almost everywhere on Σ or on Γ if f has enough regularity like $f \in L^2(\Sigma)$.*

Now, we can get higher regularity results.

Theorem 5.11. *Suppose m is a nonnegative integer and assume $f \in H^m(\Sigma)$. We assume $u \in H^1_{0,E}(\Sigma)$ is a weak solution to Problem (5.1). Then*

$$u \in H^{m+2}(\Sigma)$$

with the estimate

$$\|u\|_{H^{m+2}_\theta(\Sigma)} \le C\left(\|f\|_{H^m_\theta(\Sigma)} + \|u\|_{L^2_\theta(\Sigma)}\right).$$

Proof. The proof is quite close to the standard proof in the textbooks for elliptic PDEs. So we only illustrate the idea in this proof.

By induction, we assume this theorem holds for $m \leq m_0$, and we want to prove this theorem holds for $m = m_0 + 1$, too. Now we fix a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ such that $|\alpha| := \sum_{i=1}^{n} \alpha_i = m_0 + 1$ and $\alpha_1 = 0$. We will also work at the local coordinate near $p \in \Gamma$. We will use the notation

$$D^\alpha u = \frac{\partial^\alpha}{\partial x^\alpha} u := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha^1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u.$$

First of all, we can show $\tilde{u} := D^\alpha u$ is in $H_E^1(B_{\delta+})$ and solve a weak particular problem (similar to Problem (5.1)) with $f \in L^2(\Sigma)$. So we know $D^\alpha u \in H^2(\Sigma)$ with some estimations.

In particular, we can get $\|D^\alpha u\|_{L_\theta^2(\Sigma)} \leq C(\|f\|_{H_\theta^{m_0+1}(\Sigma)} + \|u\|_{L_\theta^2(\Sigma)})$ for any $|\alpha| = m_0 + 3$ and $0 \leq \alpha_1 \leq 2$.

Now, we can do the induction on the value of α_1 to estimate $\|D^\alpha u\|_{L_\theta^2(\Sigma)}$ using the equation for u since the equation will hold point-wise almost everywhere. This step is very similar to the last step in the proof of Theorem 5.7. $\square$

At last, we can get a smooth solution if we suppose f is smooth using general Sobolev inequalities.

Theorem 5.12. *Suppose $u \in H_{0,E}^1(\Sigma)$ is a weak solution to Problem (5.1) for some smooth function $f \in C^\infty(\Sigma)$, then we know*

$$u \in C_E^{(0)}(\Sigma).$$

In particular, u is a classical solution to Problem (5.1).

5.4 Eigenvalues and eigenfunctions

In this section, we will define the eigenvalues and eigenfunctions for the symmetric elliptic operator (L, h).

Definition 5.13. We say $\lambda \in \mathbb{R}$ is an eigenvalue for the symmetric elliptic operator (L, h) if there exists a non-trivial $u \in H^1_{0,E}(\Sigma)$ such that u solves Problem (5.1) for $f = \lambda u$. We say such u is an eigenfunction corresponding to λ.

Note that by the regularity theorem (Theorem 5.11), we can improve the regularity of u to $H^m(\Sigma)$ for any natural number m by induction, and hence, u is indeed a smooth function.

By the properties of compact operators (spectral theory of compact operators), we can get the following theorem directly.

Theorem 5.14. *The set of eigenvalues A of (L, h) is at most countable. Moreover, we can write $A = \{\lambda_k\}_{k=1}^{\infty}$ counting multiplicity with $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ such that $\lambda_k \to \infty$ as $k \to \infty$.*

For each eigenvalue λ_k, there is a corresponding eigenfunction $w_k \in H^1_{0,E}(\Sigma)$ such that $\{w_k\}_{k=0}^{\infty}$ forms an orthonormal basis of $L^2(\Sigma)$ with respect to the inner product $(\cdot, \cdot)_{L^2_\theta(\Sigma)}$.

Moreover, w_k, w_l are also orthogonal to each other for $k \neq l$ with respect to the bilinear form $B[\cdot, \cdot]$.

Proof. We consider the compact operator P defined in the proof of Theorem 5.5. Note that since P is a self-adjoint, compact, injective operator on $L^2(\Sigma)$, we may find the eigenvalues $\{\sigma_k\}_{k=1}^{\infty}$ (counting multiplicity) and corresponding eigenfunctions $\{w_k\}_{k=1}^{\infty}$ such that $\sigma_k \to 0$ and $\{w_k\}_{k=1}^{\infty}$ forms an orthonormal basis of $L^2(\Sigma)$ with respect to the inner product $(\cdot, \cdot)_{L^2_\theta(\Sigma)}$. Note that none of σ_k is zero since P is an injection. We may note $\sigma_k > 0$ since

$$(P(f), f)_{L^2_\theta(\Sigma)} = B_\gamma[P(f), P(f)] \geq 0, \quad \forall f \in L^2(\Sigma).$$

At last, we have

$$\begin{aligned}
P(w_k) = \sigma_k w_k \iff B_\gamma[P(w_k), v] &= \sigma_k B_\gamma[w_k, v], && \forall v \in H^1_{0,E}(\Sigma), \\
\iff \quad (w_k, v)_{L^2_\theta(\Sigma)} &= \sigma_k B[w_k, v] + \sigma_k \gamma (w_k, v)_{L^2_\theta(\Sigma)}, && \forall v \in H^1_{0,E}(\Sigma), \\
\iff \qquad B[w_k, v] &= \left(\tfrac{1}{\sigma_k} - \gamma\right) (w_k, v)_{L^2_\theta(\Sigma)}, && \forall v \in H^1_{0,E}(\Sigma).
\end{aligned}$$

Hence w_k is the eigenfunction of (L, h), which is corresponding to the eigenvalue $\lambda_k := \frac{1}{\sigma_k} - \gamma$. We can rearrange λ_k such that λ_k is an increasing sequence such that $\lambda_k \to +\infty$.

Note that we observe

$$B[w_k, w_l] = \lambda_k (w_k, w_l)_{L^2_\theta(\Sigma)} = 0, \quad \forall k \neq l.$$

□

Standard steps can show that the eigenvalues can be characterized by a variational formula using the Rayleigh quotient as

$$\lambda_k = \min_{V_k \subset H^1_{0,E}(\Sigma)} \max_{u \in V_k} \frac{B[u,u]}{\|u\|^2_{L^2_\theta(\Sigma)}},$$

where V_k is any k-dimensional subspace of $H^1_{0,E}(\Sigma)$.

We call the first eigenvalue λ_1 as the *principal eigenvalue* of (L, h).

Remark 5.15. *Note that, in general, the principal eigenvalue λ_1 is not simple. Here is a simple example. Suppose Σ is a triple junction hypersurface without boundary (i.e., $\partial\Sigma = \emptyset$) and consider the operator $(\Delta, 0)$ on space $H^1_{0,E_\theta}(\Sigma)$ where $\theta = (1,1,1)$. It is easy to find the principal eigenvalue is 0 and the corresponding eigenfunctions are locally constant functions (the function $f = (f^1, f^2, f^3)$ that $f^i \equiv c^i$ is a constant on Σ^i).*

Note that the eigenfunction corresponding the principal eigenvalue may change signs due to the condition $\sum_{i=1}^3 u^i = 0$ on Γ if $u \in H^1_{0,E_\theta}(\Sigma)$.

Readers may find more examples regarding this part in the paper [Wan21a]. The dimension of eigenspace corresponding to the principal eigenfunction might be quite large, although they are all connected under quotient topology.

Remark 5.16. *If the elliptic operator (L, h) is defined on the space $H^1_{0,E_1}(\Sigma)$, then the principal eigenvalue is simple. Moreover, the corresponding eigenfunction for the principal eigenvalue does not change sign on Σ.*

Chapter 6

Index and nullity of symmetric elliptic operators

In this chapter, we fix a triple junction hypersurface Σ with $C^{(1)}$ metric used in Chapter 4 and Chapter 5. Besides, we will also fix a second-order symmetric elliptic operator (L, h) defined on Sobolev space $H^1_{0,E}(\Sigma)$ for $E = E_1$ or F_θ (Indeed, we can assume E is the subbundle such that the Regularity Theorem 5.7 holds). From this chapter, we will simply write L instead of (L, h) to denote the second-order symmetric elliptic operator. We will use $B[\cdot, \cdot]$ to denote the bilinear form associated with L.

We will give a method of computing index and nullity for triple junction hypersurface Σ using Dirichlet-to-Neumann maps. This method has been successfully carried out in [Wan21a] to compute the index and nullity of stationary networks.

6.1 Basic definitions

From Theorem 5.14, we know we can write $\{\lambda_k\}_{k=1}^{\infty}$ as the eigenvalues of L such that $\lambda_1 \leq \lambda_2 \leq \cdots$ with $\lambda_k \to \infty$.

Definition 6.1. We define the *index* of L, and write it as $\mathrm{Ind}(L)$, to be the largest k such that $\lambda_k < 0$.

We define the *nullity* of L, and write it as $\mathrm{Nul}(L)$, to be the number of λ_k such that $\lambda_k = 0$.

Note that by the min-max characterization of eigenvalues of L, we have the following results.

Proposition 6.2. *The index of L is the dimension of the largest subspace of $H^1_{0,E}(\Sigma)$ such that B is negative-definite on it.*

The nullity of L is the dimension of the largest subspace of $H^1_{0,E}(\Sigma)$ such that B vanishes on it.

Here, we say B vanishes on subspace $V \subset H^1_{0,E}(\Sigma)$ if for any $v \in V$ and $u \in H^1_{0,E}(\Sigma)$, we have $B[u, v] = 0$.

We define the several notations used later on.

- $\mathcal{J}_\lambda(L)$, the space of all eigenfunctions corresponding to the eigenvalue λ.
- $\mathcal{J}_0^-(L) := \bigoplus_{\lambda<0} \mathcal{J}_\lambda(L)$.
- $\mathcal{J}_0^0(L) := \mathcal{J}_0(L)$. This is the null space of L.

We can easily find $\mathrm{Ind}(L) = \dim \mathcal{J}_0^-(L)$ and $\mathrm{Nul}(L) = \dim \mathcal{J}_0^0(L)$.

6.1.1 Stability operators and stable triple junction hypersurfaces

If $\phi : \Sigma \to N$ is an extrinsic triple junction hypersurface in N, where N is an $(n+1)$-dimensional Riemannian manifold, then we can define the stability operator $J = (J, h)$ using the second variation formula (4.11).

Definition 6.3. We say $J = (J, h)$ is a *stability operator* for Σ if we define

$$J = \Delta_\Sigma + |A_\Sigma|^2 + \mathrm{Ric}^N(\nu), \quad h = -\langle \boldsymbol{H}_\Gamma, \tau \rangle_{g_N},$$

and suppose the operator (J, h) acts on space $H^1_{0,\theta}(\Sigma)$.

Now we can define the Morse index and nullity for triple junction hypersurface Σ.

Definition 6.4. We define the (Morse) index of Σ, written as $\mathrm{Ind}(\Sigma)$, as the index of J. Similarly, we define the nullity of Σ, written as $\mathrm{Nul}(\Sigma)$, as the nullity of J.

In particular, we can talk about the stable triple junction hypersurfaces.

Definition 6.5. We say Σ is *stable* if $\mathrm{Ind}(\Sigma) = 0$.

6.1.2 Curvature estimate and generalized Bernstein theorem for triple junction surfaces

In this subsection, we will briefly summarize the result in [Wan22]. Usually, the curvature estimate will become complicated due to the terms on triple junctions.

Let us assume $\phi : \Sigma \to \mathbb{R}^3$ is a minimal triple junction surface in $\mathbb{R}^3$ equipped with the standard Euclidean metric. We will suppose Σ is stable.

From Schoen, Simon, Yau's trick [SSY75], we can get the L^p curvature estimate for triple junction surface Σ.

Theorem 6.6. *[Wan22, Theorem 5] Suppose Σ is a stable minimal triple junction surface in $\mathbb{R}^3$. Let $f \in H_0^1(\Sigma) \cap L^\infty(\Sigma)$ such that $\mathrm{sign}(f)\,|A_\Sigma|^{p-1}\,|f|^p$ is a section of F_θ almost everywhere when restricted on Γ. Then for any $1 < p < \frac{5}{4}$, we have*

$$\begin{aligned}\int_\Sigma |A_\Sigma|^2\,|f|^{2p}\,\theta d\Sigma \le C\int_\Sigma |A_\Sigma|^{2p-2}\,|f|^{2p-2}\,|\nabla_\Sigma f|^2\,\theta d\Sigma \\ + \int_\Gamma\left[\frac{p-1}{2}\left|\frac{\partial}{\partial\tau}(\log|A_\Sigma|)\right| - \boldsymbol{H}_\Gamma\cdot\tau\right]|A_\Sigma|^{2p-2}\,|f|^{2p}\,\theta d\Gamma,\end{aligned} \tag{6.1}$$

for some constant C which does not depend on p.

Note that there is an extra boundary term on Γ as shown in inequality (6.1).

The proof of this theorem is quite similar to Schoen, Simon, and Yau's standard proof. We need to keep in mind the condition $1 < p < \frac{5}{4}$ and use some tricks to deal with the singular integration.

As a consequence of L^p curvature estimate, we can get a type of generalized Bernstein theorem.

Theorem 6.7. *[Wan22, Theorem 6] Suppose Σ is an orientable stable minimal triple junction surface in $\mathbb{R}^3$. We assume Σ is complete without boundary. Then Γ cannot be compact.*

Recall that orientable is defined in Subsection 3.1.3. Note that complete Σ may not be bounded in $\mathbb{R}^3$ and hence the stable here means the stability inequality holds for any smooth variation with compact support.

Remark 6.8. *As a corollary, we know if Σ is a complete orientable stable minimal triple junction surface in $\mathbb{R}^3$ and Γ is compact, then Σ is unstable.*

Key steps in the proof of Theorem 6.7. We only illustrate the key steps in the proof of Theorem 6.7. Interested readers can refer [Wan22] for details.

We prove this theorem by contradiction. Suppose Σ is a complete orientable stable minimal triple junction surface in $\mathbb{R}^3$ and assume Γ is compact.

First, let us consider the case of none of Σ^i is flat.

- Write L^p estimate (6.1) in the following ways,

$$\begin{aligned}
\int_\Sigma \theta \left|A_\Sigma\right|^{2p} |f|^{2p} \, d\Sigma &\le C_1 \mathrm{I} + \mathrm{II} - \mathrm{III}, \\
\mathrm{I} := &\int_\Sigma \theta \left|A_\Sigma\right|^{2p-2} |f|^{2p-2} \left|\nabla_\Sigma f\right|^2 d\Sigma, \\
\mathrm{II} := &\int_\Gamma \frac{p-1}{2}\theta \left|\frac{\partial}{\partial \tau} \log |A_\Sigma|\right| |A|^{2p-2} |f|^{2p} \, d\Gamma, \\
\mathrm{III} := &\int_\Gamma \boldsymbol{H}_\Gamma \cdot \tau\theta \left|A_\Sigma\right|^{2p-2} |f|^{2p} \, d\Gamma.
\end{aligned}$$

- Fix three nonzero constants c^i such that $\sum_{i=1}^3 \theta^i c^i = 0$. Choose a cut-off function ρ_r supported in $T_{2r}(\Gamma)$ and equal to 1 in $T_r(\Gamma)$. Here $T_r(\Gamma)$ is the tubular neighborhood of Γ with the distance induced by metric g_Σ. Define $g^i = \prod_{j\ne i} |A_{\Sigma^j}|$. Choose f as

$$f^i = \operatorname{sign}(c^i) \left|c^i\right|^{\frac{1}{p}} \left(\rho_1 (g^i)^{\frac{p-1}{p}} + \rho_r - \rho_1\right)$$

  for $r > 2$. Hence, we can check $\operatorname{sign}(f) |f|^p |A_\Sigma|^{p-1} \in H^1_\theta(\Sigma)$. (Note that $F_\theta = E_\theta$ here.)

- We can estimate $\mathrm{III} = \int_\Gamma \boldsymbol{H}_\Gamma \cdot \tau\theta \left|A_\Sigma\right|^{2p-2} |f|^{2p} \, d\Gamma = \int_\Gamma \boldsymbol{H}_\Gamma \cdot \tau c^2 \theta \prod_{i=1}^3 |A_{\Sigma^i}|^{2p-2} \, d\Gamma \ge 0$ by adjusting c^i.

- We can control $\mathrm{II} = \int_\Gamma \frac{p-1}{2}\theta \left|\frac{\partial}{\partial\tau}\log|A_\Sigma|\right| |A|^{2p-2}|f|^{2p}\,d\Gamma < \varepsilon$ by choosing p close to 1 enough.

- Now, let us estimate the term I. We write

$$\begin{aligned}\mathrm{I} &= \int_\Sigma \theta\,|A_\Sigma|^{2p-2}\,|f|^{2p-2}\,|\nabla_\Sigma f|^2\,d\Sigma \\ &= \left(\int_S + \int_{T_2(\Gamma)\setminus S} + \int_{T_{2r}(\Gamma)\setminus T_r(\Gamma)}\right)\theta\,|A_\Sigma|^{2p-2}\,|f|^{2p-2}\,|\nabla_\Sigma f|^2\,d\Sigma,\end{aligned}$$

  and split $\mathrm{I} = \mathrm{I}'$(Singularity) $+\mathrm{I}_1$(Regular and near Γ) $+\mathrm{I}_2$(Regular and far from Γ). Here, S denotes the small neighborhood of zeros of second fundamental form. (Recall that the set of zeros of second fundamental form is discrete if Σ^i is non-flat.) We can choose the region S small to ensure $\mathrm{I}' < \varepsilon$. Then we choose p closed to 1 enough to ensure $\mathrm{I}_1 < \varepsilon$. At last, we fix p and choose r large enough to ensure $\mathrm{I}_2 < \varepsilon$.

Combining the above inequalities, we can find $|A_\Sigma|$ should vanish everywhere. Hence, we know at least one of Σ^i is flat. Let us suppose Σ^1 is flat, then we can find that Σ^2 and Σ^3 are capillary minimal hypersurfaces in some half-spaces. We can use the similar method to get $A_{\Sigma^i} \equiv 0$ for $i = 2, 3$.

Note that if Σ_1 is flat, we can find Σ_2, Σ_3 are minimal surfaces with constant contact angle with a plane in $\mathbb{R}^3$. There are several results related to the curvature estimates for capillary minimal surfaces. See for instance [HS21, LZ21].

Now if we know all of Σ_i are flat, we know the intersection of Σ_1 and Σ_2 should be a part of straight line. Hence, by the unique contiunation and completeness of minimal surfaces, we know Γ contains a straight line. This contradicts the fact that Γ is compact. □

Remark 6.9. *Although Theorem 6 in [Wan22] needs the condition that Σ has quadratic area growth, it has been pointed out by Luen-fai Tam that we can deduce Σ has quadratic area growth from the stable condition and Γ being compact by the argument due to D. Fischer-Colbrie [FC85].*

The tough part for the proof of Theorem 6.7 is, we need the careful choice of the test function f in the L^p curvature estimate (6.1) such that f will satisfy the condition in Theorem 6.6. Usually, the constant functions do not meet those conditions. This will introduce many extra terms near Γ, and we can analyze them carefully to ensure they are all controlled. In addition, we also need to choose p sufficiently closed to 1 to control the boundary terms. This is why we only get the generalized Bernstein theorem only for the surface case and why we require Γ to be compact.

Remark 6.10. *There is another proof for Theorem 6.7. We assume Σ as stated in Theorem 6.7 and Γ compact. Let's fix three nonzero constants c^i with $\sum_{i=1}^{3}\theta^i c^i = 0$. For any positive integer n, we can choose the test function $f \in H^1_\theta(\Sigma)$ as*

$$f^i(p) = \begin{cases} c^i, & \operatorname{dist}(p,\Gamma) \le 1 \\ c^i\left(1 - \frac{\log \operatorname{dist}(p,\Gamma)}{n}\right), & 1 \le \operatorname{dist}(p,\Gamma) \le e^n, \\ 0, & \operatorname{dist}(p,\Gamma) \ge e^n. \end{cases}$$

Put f into the stability inequality, and we have

$$\int_{\Sigma\cap T_1(\Gamma)} c^2 |A|^2 \theta d\Sigma \le \int_\Sigma |\nabla f|^2 d\Sigma - \int_\Gamma c^2 \boldsymbol{H}_\Gamma \cdot \tau\theta d\Sigma.$$

We can choose a suitable c to make sure $\int_\Gamma c^2 \boldsymbol{H}_\Gamma \cdot \tau\theta d\Gamma \ge 0$ as before. Then, we have

$$\int_{\Sigma\cap T_1(\Gamma)} c^2 |A|^2 \theta d\Sigma \le \sum_{i=1}^{n} \int_{\Sigma\cap\left(T_{e^i}(\Gamma)\setminus T_{e^{i-1}}(\Gamma)\right)} \frac{c^2}{e^{2i-2}n^2}\theta d\Sigma \le \frac{C}{n}\sum_{i=1}^{3}(c^i)^2.$$

Hence, we can choose $n \to \infty$ to get $|A| \equiv 0$ in $T_1(\Gamma)$. This impies Σ^i is flat for each i.

6.2 Dirichlet-to-Neumann maps and index theorem

6.2.1 Dirichlet-to-Neumann maps on each Σ^i

We will fix an operator L on triple junction hypersurface Σ.

For any $i = 1,2,3$, we will define a Dirichlet-to-Neumann map T^i on boundary $\mathring{\Gamma}^i = \Gamma$.

We use the following notations.

- $\mathcal{J}_\lambda(\Sigma^i, L)$, the space of all eigenfunctions corresponding to the eigenvalue λ of operator L defined on Σ^i. More precisely, it is the space of functions solving the following problem,

$$\begin{cases} -Lu = \lambda u, & \text{in } \Sigma^i, \\ u = 0, & \text{on } \partial\Sigma^i. \end{cases}$$

- $\mathcal{J}_0^-(\Sigma^i, L) := \bigoplus_{\lambda<0} \mathcal{J}_\lambda(\Sigma^i, L)$.
- $\mathcal{J}_0^0(\Sigma^i, L) := \mathcal{J}_0(\Sigma^i, L)$.
- $D_\tau \mathcal{J}_0^0(\Sigma^i, L) := \{J(\nabla v, \tau^i) \in C^\infty(\Gamma) : v \in \mathcal{J}_0^0(\Sigma^i, L)\}$, where J is the symmetric 2-tensor associated with the operator L.
- $(D_\tau \mathcal{J}_0^0(\Sigma^i, L))^\perp := \{v \in C^\infty(\Gamma) : \int_\Gamma vwd\Gamma = 0, \forall w \in D_\tau \mathcal{J}_0^0(\Sigma^i, L)\}$. This is the L^2 orthogonal complement space of $D_\tau \mathcal{J}_0^0(\Sigma^i, L)$.
- $\mathrm{Ind}(\Sigma^i, L) = \dim \mathcal{J}_0^-(\Sigma^i, L)$, the index of Σ^i with respect to the operator L.
- $\mathrm{Nul}(\Sigma^i, L) = \dim \mathcal{J}_0^0(\Sigma^i, L)$, the nullity of Σ^i with respect to the operator L.

Remark 6.11. *Note that all these definitions are only related to the operator $L^i := L|_{\Sigma^i}$.*

For any $g \in C^\infty(\Gamma)$, we consider the following problem on Σ^i.

$$\begin{cases} Lu = 0, & \text{in } \Sigma^i, \\ u = g, & \text{on } \Gamma, \\ u = 0, & \text{on } \Gamma^i. \end{cases} \tag{6.2}$$

The classical existence theorem for elliptic PDEs on Σ^i implies the following results.

Proposition 6.12. *Problem (6.2) has a smooth solution if and only if*

$$g \in (D_\tau \mathcal{J}_0^0(\Sigma^i, L))^\perp.$$

Moreover, this solution is unique up to an addition of function $v \in \mathcal{J}_0^0(\Sigma^i, L)$.

Based on Proposition 6.12, we may define the Dirichlet-to-Neumann through the following proposition.

Proposition 6.13. *For each $g \in (D_\tau \mathcal{J}_0^0(\Sigma^i, L))^\perp$, we can choose a solution u_g to Problem (6.2) with given g such that*

$$J(\nabla u_g, \tau)\big|_\Gamma + h^i g \in (D_\tau \mathcal{J}_0^0(\Sigma^i, L))^\perp.$$

Moreover, such u_g is unique, and hence we can define the Dirichlet-to-Neumann map T^i by

$$\begin{aligned} T^i : (D_\tau \mathcal{J}_0^0(\Sigma^i, L))^\perp &\rightarrow (D_\tau \mathcal{J}_0^0(\Sigma^i, L))^\perp \\ g &\rightarrow J(\nabla u_g, \tau)\big|_\Gamma + h^i g. \end{aligned}$$

Recall that h is the function defined on Γ associated with elliptic operator L. We will call u_g as the *L-extension* of g.

Proof of Proposition 6.13. Let u be one solution to Problem (6.2) with given g. First of all, we can find a unique $g_0 \in D_\tau \mathcal{J}_0^0(\Sigma^i, L)$ such that

$$J(\nabla u, \tau)|_\Gamma + h^i g + g_0 \perp D_\tau \mathcal{J}_0^0(\Sigma^i, L)$$

by the properties of orthogonal projection. Let $v_0 \in \mathcal{J}_0^0(\Sigma^i, L)$ such that $J(\nabla v_0, \tau) = g_0$ on Γ. Then we can choose $u_g = u + v_0$ and note u_g is a solution to Problem (6.2) by Proposition 6.13.

If there is another solution $\tilde{u}_g$ satisfying the same conditions, we can consider $u = u_g - \tilde{u}_g$, which solves

$$\begin{cases} Lu = 0, & \text{on } \Sigma^i, \\ u = 0, & \text{on } \partial\Sigma^i, \\ J(\nabla u, \tau) = 0, & \text{on } \Gamma. \end{cases} \tag{6.3}$$

The third condition in Problem (6.3) comes from the following reasons.

Note $u \in \mathcal{J}_0^0(\Sigma^i, L)$ by the first two conditions in Problem (6.3). So $J(\nabla u, \tau)|_\Gamma \in D_\tau \mathcal{J}_0^0(\Sigma^i, L)$. On the other hand, we know $J(\nabla u_g, \tau)\big|_\Gamma + h^i g \in (D_\tau \mathcal{J}_0^0(\Sigma^i, L))^\perp$, $J(\nabla \tilde{u}_g, \tau)\big|_\Gamma + h^i g \in (D_\tau \mathcal{J}_0^0(\Sigma^i, L))^\perp$, hence $J(\nabla u, \tau)|_\Gamma \in (D_\tau \mathcal{J}_0^0(\Sigma^i, L))^\perp$. This will imply $J(\nabla u, \tau) = 0$ on Γ.

However, by the Hopf lemma, we should have $u \equiv 0$ on Σ^i. Hence, u_g is unique. So the Dirichlet-to-Neumann map T^i is well-defined. $\square$

Note that T^i is a self-adjoint compact operator defined on some function spaces on Γ according to the results in [AtEKS14]. Moreover it has discrete eigenvalues and we write them as $\sigma_1(T^i) \leq \sigma_2(T^i) \leq \cdots$ such that $\sigma_j(T^i) \to \infty$ as $j \to \infty$. These eigenvalues are usually called Steklov eigenvalues.

Remark 6.14. *The usual Dirichlet-to-Neumann map defined on Σ^i won't contain the h^i term here. We add h^i term since we will use it to define the Dirichlet-to-Neumann on Σ. Note that it is a self-adjoint operator defined on $(D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp$. Such an operator remains compact since h^i is smooth and bounded.*

Moreover, the operator T^i here gives us the connection between the following two eigenvalue problems. The first problem is the problem with zero Dirichlet boundary condition on Σ^i,

$$\begin{cases} -Lu = \lambda u, & \text{in } \Sigma^i, \\ u = 0, & \text{on } \partial\Sigma. \end{cases} \tag{6.4}$$

The second one is the problem with Neumann boundary condition associated with h^i defined as

$$\begin{cases} -Lu = \lambda u, & \text{in } \Sigma^i, \\ u = 0, & \text{on } \Gamma^i, \\ J(\nabla u, \tau) + h^i u = 0, & \text{on } \Gamma. \end{cases} \tag{6.5}$$

Then the index and nullity of Problem (6.5) can be computed by the index and nullity of (6.4) and T^i. Precisely, we have the following results,

- $\mathrm{Ind}(N) = \mathrm{Ind}(D) + \mathrm{Ind}(T^i) + \mathrm{Nul}(D)$,
- $\mathrm{Nul}(N) = \mathrm{Nul}(T^i)$,

where we use D to denote Problem (6.4), and N to denote Problem (6.5).

Readers may refer to H. Tran's work [Tra20] for similar works. Note that this result is slightly more general than the result in [Tra20, Theorem 3.3, Theorem 3.6] since we do not require h^i to be a *constant. The proof for the index and nullity theorem for this case is essentially the same as his work with obvious modification.*

6.2.2 Dirichlet-to-Neumann maps on triple junction hypersurface Σ

We write

- $D_\tau \mathcal{J}_0^0(\Sigma, L) := \{g = (g^1, g^2, g^3) \in C^\infty(\Gamma, \mathbb{R}^3) : g^i \in D_\tau \mathcal{J}_0^0(\Sigma^i, L), \forall i = 1, 2, 3\}$,
- $(D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp := \{g = (g^1, g^2, g^3) \in C^\infty(\Gamma, \mathbb{R}^3) : g^i \in (D_\tau \mathcal{J}_0^0(\Sigma^i, L))^\perp, \forall i = 1, 2, 3\}$.

It is easy to see that $C^\infty(\Gamma, \mathbb{R}^3) = D_\tau \mathcal{J}_0^0(\Sigma, L) \oplus (D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp$ is an orthogonal decomposition of $C^\infty(\Gamma, \mathbb{R}^3)$ with respect to the inner product of $L^2_\theta(\Gamma)$. (Recall that $(g_1, g_2)_{L^2_\theta(\Gamma)} = \sum_{i=1}^3 \int_\Gamma g_1^i g_2^i \theta^i d\Gamma$.)

By the definition of Dirichlet-to-Neumann maps on each Σ^i, we know we can define a map

$$\begin{aligned} \overline{T} : (D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp &\to (D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp, \\ g = (g^1, g^2, g^3) &\to \overline{T}(g) = (T^1(g^1), T^2(g^2), T^3(g^3)). \end{aligned}$$

In general, we are interested in the function space $C_E^{(0)}(\Sigma)$. Note that we have the orthogonal decomposition of $C^\infty(\Sigma, \mathbb{R}^3)$ as

$$C^\infty(\Sigma, \mathbb{R}^3) = C_E^{(0)}(\Sigma) \oplus C_{E^\perp}^{(0)}(\Sigma).$$

Hence, we define the orthogonal projection P_E as

$$\begin{aligned} P_E : C^\infty(\Sigma, \mathbb{R}^3) &\to C_E^{(0)}(\Sigma), \\ g &\to P_E(g). \end{aligned}$$

Now, we can define the Dirichlet-to-Neumann map T on the set $C_E^{(0)}(\Sigma) \cap (D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp$ by $T(g) = P_E \circ \overline{T}(g)$. Note that the projection P_E will map $(D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp$ to the set $(D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp \cap C_E^{(0)}(\Sigma)$ by the property of projection. Hence T is an operator on the space $V_{E,L}(\Sigma)$ where we define

$$V_{E,L}(\Sigma) := (D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp \cap C_E^{(0)}(\Sigma). \tag{6.6}$$

Remark 6.15. *Note that $(D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp$ has an orthogonal decomposition $(D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp = V_{E,L}(\Sigma) \oplus V_{E^\perp,L}(\Sigma)$.*

It is easy to verify that T is a compact self-adjoint operator on the space $V_{E,L}(\Sigma)$. Hence, it will have discrete eigenvalues. We write $\sigma_k(T)$ as the eigenvalues of T where we assume $\sigma_1 \leq \sigma_2 \leq \cdots \to \infty$. (See Remark 6.16) For each $\sigma \in \mathbb{R}$, we let G_σ be the subspace of $V_{E,L}(\Sigma)$ with eigenvalue σ with respect to the operator T.

We use the following notations similarly as before.

- $G^-(T) := \bigoplus_{\sigma<0} G_\sigma$,
- $G^0(T) := G_0(T)$,
- $\mathrm{Ind}(T) := \dim G^-(T)$,
- $\mathrm{Nul}(T) := \dim G^0(T)$.

We call the operator T defined on $V_{E,L}(\Sigma)$ as the *Dirichlet-to-Neumann map* on triple junction hypersurface Σ. Note that the k-th eigenvalue of T can also be characterized variationally by

$$\sigma_k(T) = \min_{V_k \in V_{E,L}(\Sigma)} \max_{g \in V_k} \frac{(T(g), g)_{L^2_\theta(\Gamma)}}{(g, g)_{L^2_\theta(\Gamma)}} \tag{6.7}$$

where V_k is the k-dimensional subspace of $V_{E,L}(\Sigma)$.

Remark 6.16. *From the min-max identity (6.7), we can find $\sigma_1(T) \geq \min\left\{\sigma_1(T^1), \sigma_1(T^2), \sigma_1(T^3)\right\}$ as we note for any $g \in V_{E,L}(\Sigma)$, we have*

$$(T(g), g)_{L^2_\theta(\Gamma)} = (P_E(\overline{T}(g)), g)_{L^2_\theta(\Gamma)} = (\overline{T}(g), g)_{L^2_\theta(\Gamma)} = \sum_{i=1}^{3} \theta^i (T^i(g^i), g^i)_{L^2(\Gamma)}$$

$$\geq \sum_{i=1}^{3} \theta^i \sigma_1(T^i)(g^i, g^i)_{L^2(\Gamma)} \geq \min_{1\leq i\leq 3}\left\{\sigma_1(T^i)\right\} \sum_{i=1}^{3} \theta^i (g^i, g^i)_{L^2(\Gamma)} \geq \min_{1\leq i\leq 3}\left\{\sigma_1(T^i)\right\} (g, g)_{L^2_\theta(\Gamma)}.$$

This shows $\sigma_1(T) > -\infty$ and hence the formula (6.7) is well-defined.

Now we define the space related to the nullity as

$$W_{E,L}(\Sigma) := D_\tau \mathcal{J}_0^0(\Sigma, L) \cap C_E^{(0)}(\Sigma).$$

Then, the space $C^\infty(\Sigma, \mathbb{R}^3)$ can be decomposed as

$$C^\infty(\Sigma, \mathbb{R}^3) = V_{E,L}(\Sigma) \oplus W_{E,L}(\Sigma) \oplus V_{E^\perp,L}(\Sigma) \oplus W_{E^\perp,L}(\Sigma).$$

Then, we have the following index theorem for triple junction hypersurface Σ.

Theorem 6.17 (Index Theorem for triple junction hypersurface Σ). *The index of elliptic operator of L defined on space $C_E^{(0)}(\Sigma)$ can be computed by*

$$\mathrm{Ind}(L) = \sum_{i=1}^{3} \mathrm{Ind}(\Sigma^i, L) + \mathrm{Ind}(T) + \dim(W_{E,L}(\Sigma)). \tag{6.8}$$

Similarly, the nullity of L can be computed by

$$\mathrm{Nul}(L) = \mathrm{Nul}(T) + \dim(W_{E^\perp,L}(\Sigma)). \tag{6.9}$$

Proof. This proof is essentially the same as Index Theorem for Networks [Wan21a, Theorem 4.1] and Nullity Theorem for Networks [Wan21a, Theorem 4.3]. Actually, the case for triple junction hypersurfaces is a bit easier since we only have three hypersurfaces here. Hence, we only illustrate the key steps in this proof. Interested readers may find detailed proof in [Wan21a].

First part. Prove (6.8).

First, we prove $\mathrm{Ind}(L) \geq \sum_{i=1}^{3} \mathrm{Ind}(\Sigma^i, L) + \mathrm{Ind}(T) + \dim(W_{E,L}(\Sigma))$.

Let $\mathcal{J}_0^-(\Sigma, L) := \oplus_{i=1}^{3} \mathcal{J}_0^-(\Sigma^i, L)$ and $V^- := \{f \in C^\infty(\Sigma) : f^i$ is the L-extension of g^i for $i = 1, 2, 3$ on Σ^i for some $g = (g^1, g^2, g^3) \in G^-\}$.

A short calculation shows, B is negative-definite on $\mathcal{J}_0^-(\Sigma, L)$ and V^-. Now we can construct a new space $V_1 \subset C^\infty(\Sigma)$ based on $W_{E,L}(\Sigma)$ by the following lemma.

Lemma 6.18 (Lemma 4.2 in [Wan21a]). *Let $g \in W_{E,L}(\Sigma)$ and g is not identically zero. Define*

$$W_g := \left\{ f \in C_E^{(1)}(\Sigma) : f|_\Gamma = cg \text{ for some } c \in \mathbb{R},\ f \perp_B \mathcal{J}_0^-(\Sigma, L) \right\}.$$

Here, we use $\perp_B$ to denote the orthogonal with respect to the bilinear form B. Then L has index 1 on the space W_g.

Now, we can choose a basis $\{g_j\}_{j=1}^{l}$ of $W_{E,L}(\Sigma)$ with $l = \dim(W_{E,L}(\Sigma))$ and extend g_j to f_j based on Lemma 6.18 such that $B[f_j, f_j] < 0$ for $j = 1, 2, \cdots, l$. Hence, we can choose the space V_1 spanned by $\{f_j\}_{j=1}^{l}$.

At last, we can verify that three spaces $\mathcal{J}_0^-(\Sigma, L), V^-$ and V_1 are orthogonal to each other under the bilinear form B. Hence $\mathrm{Ind}(L) \geq \sum_{i=1}^{3} \mathrm{Ind}(\Sigma^i, L) + \mathrm{Ind}(T) + \dim(W_{E,L}(\Sigma))$.

Second, we need to show $\mathrm{Ind}(L) \leq \sum_{i=1}^{3} \mathrm{Ind}(\Sigma^i, L) + \mathrm{Ind}(T) + \dim(W_{E,L}(\Sigma))$. We write $W = \mathcal{J}_0^-(\Sigma, L) \oplus V^- \oplus V_1$. Suppose $\overline{W}$ is one of the (maximal) subspaces of $C_E^{(0)}(\Sigma)$ such that $\dim \overline{W} = \mathrm{Ind}(L)$ and B is negative definite on it. Let $P_{\overline{W},W}$ be the orthogonal projection of $\overline{W}$ to W with respect to the bilinear form B.

Now we can show that $P_{\overline{W},W}$ is onto by the maximum property of $\overline{W}$. On the other hand, we can show $P_{\overline{W},W}$ is one-to-one using Lemma 6.18.

Hence, we have $\mathrm{Ind}(L) \leq \sum_{i=1}^{3} \mathrm{Ind}(\Sigma^i, L) + \mathrm{Ind}(T) + \dim(W_{E,L}(\Sigma))$.

Second part. Prove (6.9).

The proof of nullity is a bit easier than the above proof. We define $V^0 := \{f \in C^\infty(\Sigma) : f^i$ is the L-extension of g^i for $i = 1, 2, 3$ on Σ^i for some $g \in G^0\}$. It is easy to verify that B vanishes on V^0.

For any $g \in W_{E^\perp,L}(\Sigma)$, we know there is a function $u = (u^1, u^2, u^3)$ such that $u^i|_\Gamma \equiv 0$, $J(\nabla u^i, \tau)\big|_\Gamma \equiv g^i$ on Γ and $Lu = 0$ in Σ^i based on definition of $D_\tau \mathcal{J}_0^0(\Sigma, L)$. In particular, since $g \in C_{E^\perp}^{(0)}(\Sigma)$, we know u solves the following problem,

$$\begin{cases} -Lu = 0, & \text{in } \Sigma, \\ u = 0, & \text{on } \partial\Sigma, \\ u|_\Gamma \equiv 0 \in \Gamma(E), & \\ J(\nabla u, \tau)|_\Gamma + hu|_\Gamma = J(\nabla u, \tau)|_\Gamma = g \in \Gamma(E^\perp). & \end{cases} \tag{6.10}$$

Hence, $u \in \mathcal{J}_0^-(\Sigma, L)$. If we write $V_2 := \{u : u$ is a solution to Problem (6.10) for $g \in W_{E^\perp,L}(\Sigma)\ \}$, then B vanishes on V_2. So we find the space $W_{E^\perp,L}(\Sigma)$ will contribute to the nullity of operator L. This will give us $\mathrm{Nul}(L) \geq \mathrm{Nul}(T) + \dim(W_{E^\perp,L}(\Sigma))$.

On the other hand, by an elementary argument, if $B[f, v] = 0$ for any $v \in H_E^1(\Sigma)$ for some $f \in C^{(0)}_E(\Sigma)$, then $f \in V^0 \oplus V_2$. This implies $\mathrm{Nul}(L) \leq \mathrm{Nul}(T) + \dim(W_{E^\perp,L}(\Sigma))$.

Hence, $\mathrm{Nul}(L) = \mathrm{Nul}(T) + \dim(W_{E^\perp,L}(\Sigma))$. $\square$

Remark 6.19. *We can give a short explanation of why the index theorem can hold. In particular, we need to know the meaning of spaces $W_{E,L}(\Sigma)$. Note that for any $g \in W_{E^\perp,L}(\Sigma)$, we know there is a function $u = (u^1, u^2, u^3)$ solving Problem (6.10).*

But for $g \in W_{E,L}(\Sigma)$, we can construct u such that u solves the following problem,

$$\begin{cases} -Lu = 0, & \text{in } \Sigma, \\ u = 0, & \text{on } \partial\Sigma, \\ u|_\Gamma \equiv 0 \in \Gamma(E), & \\ J(\nabla u, \tau)|_\Gamma + hu|_\Gamma = g \in \Gamma(E). & \end{cases}$$

This function looks like only satisfies $Lu = 0$ but fails to be smooth. We can imagine that the energy $B[u,u]$ will decrease when we try to make it smooth. Hence it will contribute to the index of L.

6.3 Applications

In this section, we will give an application of the index theorem.

A typical triple junction surface in $\mathbb{S}^3$ is the union of three geodesic half spheres with a common geodesic circle as their boundaries. For example, we take

$$\Sigma^i := \left\{(x_1, x_2, x_3, x_4) \in \mathbb{S}^3 : x_1 = t\cos\zeta^i, x_2 = t\sin\zeta^i, x_3^2 + x_4^2 = 1 - t^2, t \geq 0\right\}$$

for $\zeta^i = \frac{2\pi}{3}i$. Here, we take $\theta = (1,1,1)$. So Σ will be a minimal triple junction surface in $\mathbb{S}^3$.

The stability operator $J = (J, h)$ on Σ is given by

$$Ju = \Delta_\Sigma u + \mathrm{Ric}^{\mathbb{S}^3}(\nu)u = \Delta_\Sigma u + 2, \quad h = -\langle \boldsymbol{H}_\Gamma, \tau\rangle_{g_N} \equiv 0.$$

This operator is defined on the space $C^{(0)}_E(\Sigma)$ where $E = \{(p,g) \in \Gamma \times \mathbb{R}^3 : \sum_{i=1}^3 g^i = 0\}$.

Let $\mathbb{D} := \left\{(x_2, x_3, x_4) \in \mathbb{S}^2 : x_2 \geq 0\right\}$ be the half sphere. We write $L = \Delta_{\mathbb{D}} u + 2$ to be the elliptic operator on $\mathbb{D}$. It is easy to note $\mathrm{Ind}(\mathbb{D}, L) = 0$ and $\mathrm{Nul}(\mathbb{D}, L) = 1$ on $\mathbb{D}$. Moreover, the coordinate function $x_2 \in \mathcal{J}_0(\mathbb{D}, L)$. Hence, we know $\mathrm{Ind}(\Sigma^i, J) = \mathrm{Ind}(\mathbb{D}, L) = 0$, $\mathrm{Nul}(\Sigma^i, J) = \mathrm{Nul}(\mathbb{D}, L) = 1$, and $D_\tau \mathcal{J}_0^0(\mathbb{D}, L)$ only contains constant functions.

Now, we may want to know the property of $T_{\mathbb{D}}$ defined on $\mathbb{D}$. Instead of directly computing the $T_{\mathbb{D}}$, we can use the Morse index and nullity of the geodesic sphere to help us.

Recall that the Morse index of geodesic sphere $\mathbb{S}^2$ in $\mathbb{S}^3$ is one, and the nullity of $\mathbb{S}^2$ is 3. We can cut it into two half-spheres along one geodesic on this sphere. Note that, in particular, the index and nullity theorem is not only true for triple junction hypersurface but also true for some other junctions with the other orders. We can view $\mathbb{S}^2$ as a double junction surface by identifying the boundaries of two half spheres. In this case, we can also define the Dirichlet-to-Neumann map T on $\mathbb{S}^2$, which will be identical to the usual Dirichlet-to-Neumann map $T_{\mathbb{D}}$ since two half spheres are isometric to each other. Hence, we may apply the index and nullity theorem to $\mathbb{S}^2$ to get

- $\mathrm{Ind}(\mathbb{S}^2) = 2\mathrm{Ind}(\mathbb{D}) + \mathrm{Ind}(T_{\mathbb{D}}) + \dim D_\tau \mathcal{J}_0^0(\mathbb{S}^2, J) \cap \Gamma(\hat{E})$.
- $\mathrm{Nul}(\mathbb{S}^2) = \mathrm{Nul}(\mathbb{D}) + \dim D_\tau \mathcal{J}_0^0(\mathbb{S}^2, J) \cap \Gamma(\hat{E}^\perp)$.

where $\hat{E} := \{(p, g = (g^1, g^2)) \in \partial\mathbb{D} \times \mathbb{R}^2 : g^1 + g^2 = 0\}$.

Recall that $D_\tau \mathcal{J}_0^0(\mathbb{S}^2, J) = \{(p, g) \in \partial\mathbb{D} \times \mathbb{R}^2 : g^1 = c_1, g^2 = c_2\}$ only contains constant vector-valued functions, it is easy to see

$$D_\tau \mathcal{J}_0^0(\mathbb{S}^2, J) \cap \Gamma(\hat{E}) = \left\{(p, g) \in \partial\mathbb{D} \times \mathbb{R}^2 : g^1 = -g^2 = c \in \mathbb{R}\right\},$$
$$D_\tau \mathcal{J}_0^0(\mathbb{S}^2, J) \cap \Gamma(\hat{E}^\perp) = \left\{(p, g) \in \partial\mathbb{D} \times \mathbb{R}^2 : g^1 = g^2 = c \in \mathbb{R}\right\}.$$

Hence, $\dim D_\tau \mathcal{J}_0^0(\mathbb{S}^2, J) \cap \Gamma(\hat{E}) = D_\tau \mathcal{J}_0^0(\mathbb{S}^2, J) \cap \Gamma(\hat{E}^\perp) = 1$.

Hence, we get $\mathrm{Ind}(T_{\mathbb{D}}) = 0$ and $\mathrm{Nul}(T_{\mathbb{D}}) = 2$.

Remark 6.20. *Indeed, the null space of $T_{\mathbb{D}}$ is generated by the coordinate functions x_3, x_4.*

Now, from the index and nullity theorem, we can compute the index and nullity of Σ by

- $\mathrm{Ind}(\Sigma) = 3\mathrm{Ind}(\mathbb{D}) + \mathrm{Ind}(T) + \dim D_\tau \mathcal{J}_0^0(\Sigma, J) \cap \Gamma(E)$,

- $\mathrm{Nul}(\Sigma) = \mathrm{Nul}(T) + \dim D_\tau \mathcal{J}_0^0(\Sigma, J) \cap \Gamma(E^\perp)$.

It is easy to observe that $\dim D_\tau \mathcal{J}_0^0(\Sigma, J) \cap \Gamma(E) = 2$ and $\dim D_\tau \mathcal{J}_0^0(\Sigma, J) \cap \Gamma(E^\perp) = 1$ by writing down those constant functions explicitly.

Now let us analyze the operator T defined on $(D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp \cap \Gamma(E)$. We write $V := (D_\tau \mathcal{J}_0^0(\Sigma, L))^\perp \cap \Gamma(E)$ for short. Note that for $g \in V$, we know $g^3 = -g^2 - g^1$ and $g^i \in D_\tau \mathcal{J}_0^0(\mathbb{D}, L)$. So we can express V using g^1, g^2 and the T defined on V can be written as

$$T((g^1, g^2, -g^1 - g^2)) = (T_{\mathbb{D}}(g^1), T_{\mathbb{D}}(g^2), -T_{\mathbb{D}}(g^1) - T_{\mathbb{D}}(g^2)).$$

Hence, we can find $\mathrm{Ind}(T) = 2\mathrm{Ind}(T_{\mathbb{D}}) = 0$, $\mathrm{Nul}(T) = 2\mathrm{Nul}(T_{\mathbb{D}}) = 4$.

So we get $\mathrm{Ind}(\Sigma) = 2, \mathrm{Nul}(\Sigma) = 5$.

Hence, we can find that the locally constant functions contribute to the index of Σ, and every function in the null space of Σ with respect to operator $\Delta + 2$ is generated by a rotation on $\mathbb{S}^3$.

Chapter 7

Conformal structures on triple junction surfaces

In this chapter, we will only focus on the 2-dimensional triple junction hypersurfaces, namely triple junction surfaces.

7.1 Basic definition

Let $J = (J^1, J^2, J^3)$ be the almost complex structure on Σ. This means each J^i will be an almost complex structure on Σ^i. Recall that the almost complex structure means each J^i is a smooth tensor field of degree $(1,1)$ such that $(J^i)^2 = -\mathrm{Id}$ where we view $J^i : T\Sigma^i \to T\Sigma^i$ as a vector bundle homeomorphism.

Definition 7.1. We say J is *compatible* with (τ, D) if there exists a vector field $\eta \in \mathfrak{X}(\Gamma)$ such that $\tau(p) = \mathrm{sign}(p) J(\eta)(p)$ for any $p \in \Gamma$.

Recall that sign is a sign function defined in Subsection 3.1.3. Hence, this compatible condition requires that the orientation determined by J agree with the usual orientation on Σ.

Remark 7.2. *Note that the compatible condition is only depending on the equivalence class of (τ, D). That is, if we suppose $(\tau, D) \sim (\tilde{\tau}, \tilde{D})$, then J is compatible with (τ, D) if and only if J is compatible* *with* $(\tilde{\tau}, \tilde{D})$.

Hence, for any $C^{(0)}$ *triple junction surface with given almost complex structure* J *on* Σ*, if the orientation on* Σ *is induce by* Σ *(by changing the orientation on* Σ *if needed), then we can define* $C^{(1)}$ *triple junction structure by choose* $\tau = J(\eta)$ *for some non-vanishing vector field* $\eta \in \mathfrak{X}(\Gamma)$ *to ensure* $J(\eta)$ *pointing outside.*

Another way to define the conformal structure is to use the equivalence class of metrics. Recall that we have known the metric space $\mathrm{Met}^{(1)}(\Sigma)$ is invariant under conformal transformation (Proposition 3.32).

Definition 7.3. We say two metrics $g_1, g_2 \in \mathrm{Met}^{(1)}(\Sigma)$ are *conformally equivalent* to each other if there exists $h \in C^{(1)}_{E_1,E_\theta}(\Sigma)$ such that $g_2 = hg_1$.

Note that for any $g \in \mathrm{Met}^{(1)}(\Sigma)$, we can define the conformal structure J by define $J^i(e_1) = e_2$ for any orthonormal basis $\{e_1, e_2\}$ of $T_q\Sigma^i$ that $\{e_1, e_2\}$ agrees with the orientation on Σ^i.

From the standard result for conformal metrics, we have

Proposition 7.4. *g_1, g_2 are conformally equivalent to each other if and only if they induce the same conformal structure on* Σ.

Before the proof, let us recall a standard result of conformal changing of metric.

Lemma 7.5. *Let* Σ *be a Riemann surface with boundary and suppose* g *is a metric. We write* K_g *as the Gaussian curvature of* g *and* κ_g *as the geodesic curvature of* $\partial\Sigma$ *under metric* g*. Let* $u \in C^\infty(\Sigma)$*, then the Gaussian curvature and geodesic curvature can be characterized by*

$$K_{e^{2u}g} = e^{-2u}\left(K_g - \Delta_{\Sigma,g} u\right), \tag{7.1}$$

$$\kappa_{e^{2u}g} = e^{-u}\left(\kappa_g + \frac{\partial u}{\partial \tau}\right). \tag{7.2}$$

where $\Delta_{\Sigma,g}$ *is the Laplacian operator on* Σ *with respect to the metric* g.

Recall that in general, we will define the geodesic curvature as $\kappa_g = \langle \nabla_\eta \eta, -\tau \rangle$ where η is the unit tangent vector field on Γ.

Proof of Proposition 7.4. If g_1, g_2 are conformally equivalent, then we can easy to see they will induce the same conformal structure.

Conversely, if they induce the same conformal structure, then we know we can find $g_1 = hg_2$ for some $h \in C^\infty(\Sigma)$. We only need to show $h \in C^{(1)}_{E_1,E_\theta}(\Sigma)$. Note that $g_1, g_2 \in \mathrm{Met}^{(0)}(\Sigma)$ implies $h \in C^{(0)}_{E_1}(\Sigma)$.

Now we note $\sum_{i=1}^3 \theta^i \kappa^i_{g_1} = 0$ and $\sum_{i=1}^3 \theta^i \kappa^i_{g_2} = 0$, where we have used κ^i_g to denote the geodesic curvature of Γ under the metric g. Hence, by Lemma 7.5, we have

$$\frac{\partial \log h}{\partial \tau}|_\Gamma \in \Gamma(E_\theta),$$

which implies $\frac{\partial h}{\partial \tau}|_\Gamma \in \Gamma(E_\theta)$. Hence $h \in C^{(1)}_{E_1,E_\theta}(\Sigma)$. □

Conversely, we have the following proposition.

Proposition 7.6. *Suppose J is a conformal structure on Σ, then we can find $g \in \mathrm{Met}^{(1)}(\Sigma)$ such that J can be induced by g.*

Proof of Proposition 7.6. At first, we can choose g^i_0 for each i such that J^i can be induced by g^i. Now we want to find a function $h \in C^\infty(\Sigma)$ such that $hg_0 \in \mathrm{Met}^{(1)}(\Sigma)$.

Let us fix a metric g_Γ on Γ. So we know $g^i_0|_\Gamma = f^i g_\Gamma$ for some smooth and positive function f^i defined on Γ. Now we extend f^i to a smooth positive function on Σ^i and define $g_1 = \frac{1}{f} g_0$. So the g_1 will satisfy $g^i_1|_\Gamma = g_\Gamma$ on Γ.

Now let us choose $u \in C^\infty(\Sigma)$ such that $u \equiv 0$ on Γ and $\frac{\partial u^i}{\partial \tau} = -\kappa^i_{g^i_1}$. Then we can choose $g = e^{2u} g_1$.

Hence, by Lemma 7.5, we know $\kappa^i_g \equiv 0$ on Γ for each i. Note that J can also be induced by g.

Now we only need to show $g \in \mathrm{Met}^{(1)}(\Sigma)$ for some (τ, D) compatible with J. From Proposition 3.27, we know $g \in \mathrm{Met}^{(1)}(\Sigma)$ for some (τ, D) and g is the canonical metric under such triple junction structure. Hence, τ should be the outer unit normal vector field. Let η be the unit tangent vector field along Γ under metric g and agree with the orientation on Γ. Note that $J(\eta) = \mathrm{sign}\tau$ by the definition of sign and the properties of metric g, we know

(τ, D) will be compatible with J. So the metric g is one of the metrics we want. (Note that we have written $\text{sign}\tau = \text{sign}(\cdot)\tau(\cdot)$ as a multiplication of function and a vector field.) $\square$

Remark 7.7. *We may only use the equivalence class of* $\text{Met}^{(0)}(\Sigma)$ *to define the conformal structure on* Σ*. But usually, the metric in* $\text{Met}^{(1)}(\Sigma)$ *might have better properties. For example, we may have a generalized Gauss-Bonnet formula that holds on* Σ*.*

Lemma 7.8 (Generalized Gauss-Bonnet formula)**.** *Suppose* $g \in \text{Met}^{(1)}(\Sigma)$*, then we have*

$$\sum_{i=1}^{3} \int_{\Sigma^i} K_{g^i} \theta^i d\Sigma^i + \sum_{i=1}^{3} \int_{\Gamma^i} \kappa_g^i \theta^i d\Gamma^i = \sum_{i=1}^{3} \theta^i \chi(\Sigma^i).$$

Proof. This is a consequence of usual Gauss-Bonnet formula on each Σ^i and the condition $\sum_{i=1}^{3} \theta^i \kappa_{g^i}^i = 0$. $\square$

Note that if $\theta = (1, 1, 1)$ and $\partial\Sigma = \emptyset$, then we have

$$\int_{\Sigma} K d\Sigma = \chi(\Sigma)$$

where $\chi(\Sigma) = \sum_{i=1}^{3} \chi(\Sigma^i)$. For the later applications, we write $\chi(\Sigma) = \sum_{i=1}^{3} \theta^i \chi(\Sigma^i)$, too.

7.2 Uniformization

Usually, the conformal structure on surfaces plays a vital role in the studying of minimal surfaces; see, for instance, [MP04]. In order to study the conformal structure on a triple junction surface, it is better to look for a kind of uniformization on Σ at first and find a good metric that can represent this conformal structure.

7.2.1 $\chi(\Sigma) < 0$ case

Let us focus on the $\chi(\Sigma) < 0$ case first.

Theorem 7.9. *If Σ is a compact surface with $\partial\Sigma = \emptyset$ satisfying $\chi(\Sigma) < 0$, with given Riemannian metric $g \in \mathrm{Met}^{(1)}(\Sigma)$, then for any negative function $K \in C^\infty(\Sigma)$, we can find a new metric $\overline{g}$ conformally equivalent to g such that the Gaussian curvature of $\overline{g}$ is given by K.*

Note that by Lemma 7.5, the above theorem is equivalent to finding a smooth function $u \in C^{(1)}_{E_1,E_\theta}(\Sigma)$ such that

$$\Delta u = K_0 - Ke^{2u}, \tag{7.3}$$

where $K_0 = K_g$ and we use $\Delta := \Delta_{\Sigma,g}$. Hence, we will look for a solution to the following problem

$$\begin{cases} \Delta u = K_0 - Ke^{2u}, & \text{in } \Sigma, \\ u|_\Gamma \in \Gamma(E_1), & \text{on } \Gamma, \\ \frac{\partial u}{\partial \tau}|_\Gamma \in \Gamma(E_\theta), & \text{on } \Gamma. \end{cases} \tag{7.4}$$

Comparing with Problem (5.1) (without boundary terms), we can call Problem (7.4) as the semilinear elliptic problem.

The proof for Theorem 7.9 is quite standard. We may borrow the method in [Tay11, Section 2, Chapter 14].

Let us define an energy $\mathcal{E}(u)$ by

$$\mathcal{E}(u) = \int_\Sigma \left[\frac{1}{2}|\nabla u|^2 + K_0 u\right]\theta d\Sigma.$$

We consider the subset of $H^1_1(\Sigma)$ (recall that $H^1_1(\Sigma) := W^{k,2}_{E_1}(\Sigma)$) defined by

$$S := \left\{u \in H^1_1(\Sigma) : \int_\Sigma Ke^{2u}\theta d\Sigma = 2\pi\chi(\Sigma)\right\}.$$

The idea is that we want to minimize the energy $\mathcal{E}(u)$ in the class S and hope the minimizer would be a solution to Problem (7.4).

Roughly speaking, we will show the following things,

- S is a C^1-submanifold of $H^1_1(\Sigma)$.
- $\inf_S \mathcal{E} > -\infty$.

- F has a minimizer.

- The minimizer would solve Problem (7.4).

Note that $u \in S$ is a necessary condition to Theorem 7.9 since, the new metric $e^{2u}g$ will satisfy the Gauss-Bonnet formula (cf. Lemma 7.8), then we have

$$2\pi\chi(\Sigma) = \int_\Sigma \theta K e^{2u} d\Sigma.$$

Lemma 7.10. *S is a nonempty C^1-submanifold of $H^1(\Sigma)$ if $K < 0$ and $\chi(\Sigma) < 0$.*

Proof. S is nonempty since we may choose u is a constant function defined by

$$u \equiv \frac{1}{2} \log \frac{2\pi\chi(\Sigma)}{\int_\Sigma K\theta d\Sigma},$$

which is well-defined since $K < 0$ and $\chi(\Sigma) < 0$.

Now let us consider the map $J : H_1^1(\Sigma) \to \mathbb{R}$ defined on

$$J(u) = \int_\Sigma K e^{2u} \theta d\Sigma.$$

Note that we already know J is a C^1 function on $H^1(\Sigma)$ (cf. [Tay11, Lemma 2.2, Chapter 14]). Moreover, we note $DJ(u) = 2Ke^{2u}$ for any $u \in H_1^1(\Sigma)$, which is non-zero as an element of dual space of $H_1^1(\Sigma)$. Hence, by implicit function theorem, we know S is a C^1-submanifold of $H_1^1(\Sigma)$. □

Lemma 7.11. *If we have $\chi(\Sigma) < 0$, $K < 0$, then $\inf_S \mathcal{E} > -\infty$.*

Proof. Let $u_0 = u - \overline{u}$ where $\overline{u} := \frac{\int_\Sigma u\theta d\Sigma}{\int_\Sigma \theta d\Sigma}$, which is the weighed average of u. Since $u \in S$, we have

$$e^{2\overline{u}} \int_\Sigma K e^{2u_0} \theta d\Sigma = 2\pi\chi(\Sigma) \Longleftrightarrow \overline{u} = \frac{1}{2} \log \frac{2\pi\chi(\Sigma)}{\int_\Sigma K e^{2u_0} \theta d\Sigma}. \tag{7.5}$$

Hence, we have

$$
\begin{aligned}
\mathcal{E}(u) &= \int_\Sigma \left[\frac{1}{2}|\nabla u_0|^2 + K_0(u_0+\overline{u})\right]\theta d\Sigma \\
&= \int_\Sigma \left[\frac{1}{2}|\nabla u_0|^2 + K_0 u_0\theta\right] d\Sigma + \overline{u}\int_\Sigma K_0\theta d\Sigma \\
&= \int_\Sigma \left[\frac{1}{2}|\nabla u_0|^2 + K_0 u_0\theta\right] d\Sigma + \pi\chi(\Sigma)\log\frac{2\pi\chi(\Sigma)}{\int_\Sigma Ke^{2u_0}\theta d\Sigma},
\end{aligned} \tag{7.6}
$$

where we have used Gauss-Bonnet formula for the last equality.

Now we need to show $\log\left(-\int_\Sigma Ke^{2u_0}\theta d\Sigma\right) > -\infty$. This is easy to see since $-K = |K| \geq \delta > 0$ for δ small enough since Σ is compact, so we have

$$
\int_\Sigma |K| e^{2u_0}\theta d\Sigma \geq \delta\int_\Sigma (1+2u_0)\theta d\Sigma = \delta\int_\Sigma \theta d\Sigma. \tag{7.7}
$$

Hence $\log(-\int_\Sigma Ke^{2u_0}\theta d\Sigma) > -\infty$.

On the other hand, we may apply Poincaré inequality and Cauchy-Schwarz inequality to get

$$
\int_\Sigma K_0 u_0\theta d\Sigma \leq \frac{C}{\varepsilon} + \varepsilon\int_\Sigma |\nabla u_0|^2\theta d\Sigma. \tag{7.8}
$$

Note that the Poincaré is valid for u_0 by a simple contradiction argument.

Combining the inequality (7.6), (7.7), (7.8), we have

$$
\inf_{u\in S}\mathcal{E}(u) > -\infty.
$$

□

Now we can go back to the proof of our theorem.

Proof of Theorem 7.9. Let $u_k \in H_1^1(\Sigma)$ such that $\mathcal{E}(u_k) \to \inf_S \mathcal{E}$. We will prove they are bounded under $H_\theta^1(\Sigma)$ norm.

We can suppose $\mathcal{E}(u_k) \leq \inf_S \mathcal{E} + 1$. Note that by inequality (7.6), (7.7), and (7.8), we have

$$
\frac{1}{4}\int_\Sigma \theta|\nabla u_{k0}|^2\theta d\Sigma - C \leq \mathcal{E}(u_k) \leq \inf_S \mathcal{E} + 1, \tag{7.9}
$$

where $u_{k0} := u_k - \overline{u}_k$ for some constant C. Note that by identity (7.5), we know the average of u_k is bounded from above from the proof of Lemma 7.11. This is because the Moser-Trudinger inequality is valid for u_{k0}. Hence, in view of the inequality (7.9), we have

$$\int_\Sigma |K| e^{2u_{k0}} \theta d\Sigma \leq C < \infty.$$

So the average of $\overline{u}$ is bounded by identity (7.5).

Combining the inequality (7.9), we know u_k is bounded under the norm $H^1_\theta(\Sigma)$. So we may take a subsequence of u_k (denotes it as u_k), such that $u_k \to u$ weakly in $H^1_1(\Sigma)$.

Recall that the map $u \to e^{2u}$ is a compact map from $H^1(\Sigma)$ to $L^1(\Sigma)$, we know $e^{2u_k} \to e^{2u}$ in L^1 sense strongly. Hence $u \in S$.

On the other hand, we have $\int_\Sigma K_0 u_k \theta d\Sigma \to \int_\Sigma K_0 u \theta d\Sigma$ and $\liminf_{k\to\infty} \int_\Sigma \theta \left|\nabla u_k\right|^2 d\Sigma \geq \int_\Sigma \theta \left|\nabla u\right|^2 d\Sigma$, we have $\mathcal{E}(u) \leq \inf_S \mathcal{E}$. Hence u is a minimizer of $\mathcal{E}$ on S.

Now let us show u is a solution to Problem (7.4). Note that by Lagrange Multipliers Theorem, we can find some $\lambda \in \mathbb{R}$ such that u is a critical point for the functional $\mathcal{E}(u) - \lambda \left(\int_\Sigma K e^{2u} \theta d\Sigma - 2\pi\chi(\Sigma) \right)$.

Hence, we get

$$\int_\Sigma \left[\langle \nabla u, \nabla v \rangle + K_0 v - 2\lambda K e^{2u} v \right] \theta d\Sigma = 0, \quad \forall v \in H^1_1(\Sigma). \tag{7.10}$$

So we may use the regularity theorem (Theorem 5.7 and Theorem 5.11) to find $u \in H^2(\Sigma)$. This imply u is at least Hölder continuous and bounded on Σ. Hence $e^{2u} \in H^2(\Sigma)$. We can repeat it inductively to get $u \in C^{(0)}_{E_1}(\Sigma)$.

Let us choose $v \equiv 1$ in equation (7.10) and get

$$2\pi\chi(\Sigma) = \int_\Sigma K_0 \theta d\Sigma = 2\lambda \int_\Sigma K e^{2u} \theta d\Sigma.$$

Hence $\lambda = \frac{1}{2}$.

In summary, we know u solves Problem (7.4), and hence we finish the proof. □

In particular, we can choose $K = -1$ on Σ. This will lead to the following uniformization theorem.

Theorem 7.12 (Uniformization)**.** *Suppose Σ is a compact triple junction surface with given*

Riemannian metric $g \in \mathrm{Met}^{(1)}(\Sigma)$ such that $\partial\Sigma = \emptyset$ and $\chi(\Sigma) < 0$. Then there is a hyperbolic metric $\overline{g} \in \mathrm{Met}^{(1)}(\Sigma)$ which is conformally equivalent to g.

Remark 7.13. *Theorem 7.12 can also be summarized as follows.*

For any (Σ, g) satisfying condition stated in Theorem 7.9, we can find a new metric $\overline{g}$ on Σ such that the following holds,

- *Each $\overline{g}^i$ is conformal to g^i on Σ^i.*
- *$\overline{g}^i|_\Gamma \equiv \overline{g}^j|_\Gamma$ along Γ.*
- *Each $\overline{g}^i$ is a hyperbolic metric on Σ^i.*
- *$\sum_{i=1}^3 \theta^i \kappa^i_{\overline{g}^i} = 0$ along Γ.*

7.2.2 $\chi(\Sigma) = 0$ case

For this case, we can prove the following theorem.

Theorem 7.14. *Suppose Σ is a compact triple junction surface with given Riemannian metric $g \in \mathrm{Met}^{(1)}(\Sigma)$ such that $\partial\Sigma = \emptyset$ and $\chi(\Sigma) = 0$. Then there is a flat metric $\overline{g} \in \mathrm{Met}^{(1)}(\Sigma)$ which is conformally equivalent to g.*

Note that we say a metric g is flat if and only if its Gaussian curvature K_g is zero everywhere.

Proof of Theorem 7.14. The proof for this theorem is easier than the case of $\chi(\Sigma) < 0$. We can minimizing the energy $\mathcal{E}$ on the following set

$$S := \left\{ u \in H_1^1(\Sigma) : \int_\Sigma u\theta d\Sigma = 0 \right\}.$$

It is pretty easy to see the minimizer exists, and it solves the following problem weakly (and hence smoothly by regularity results),

$$\begin{cases} \Delta u = K_0, & \text{in } \Sigma, \\ u|_\Gamma \in \Gamma(E_1), & \text{on } \Gamma, \\ \frac{\partial u}{\partial \tau}|_\Gamma \in \Gamma(E_\theta), & \text{on } \Gamma. \end{cases}$$

Hence, the metric $e^{2u}g$ is the flat metric we want by Lemma 7.5. □

7.2.3 $\chi(\Sigma) > 0$ case

The case for $\chi(\Sigma) > 0$ is much harder than the case $\chi(\Sigma) \leq 0$. This is because the energy $\mathcal{E}$ does not have a lower bound anymore, so it is impossible to use the previous method. To my knowledge, we do not know if the uniformization holds in this case.

In general, even in the sphere case, we may need extra efforts to prove the uniformization. Readers may refer to [CY88, MT02, Str05, Mal17] for related works.

In the triple junction surface case, there are much more examples with $\chi(\Sigma) > 0$ since each component of Γ will increase the Euler characteristic. More precisely, suppose Σ is a compact triple junction surface with $\partial\Sigma = \emptyset$ and density $\theta = (1, 1, 1)$, we denote $\operatorname{genus}(\Sigma^i)$ as the genus of each Σ^i and k as the number of components of Γ. Then $\chi(\Sigma)$ can be computed by

$$\chi(\Sigma) = 3k - 2\sum_{i=1}^{3} \operatorname{genus}(\Sigma^i).$$

So $\chi(\Sigma)$ might be very large even if some of Σ^i has a non-zero genus.

Remark 7.15. *For the simplest case, it is unclear if there is only one conformal structure on the standard triple junction spheres (the union of three disks by identifying their boundary).*

From the examples in Section 7.3, we have learned that the space of conformal structure (e.g., moduli space) might be quite large (infinite dimension). So it is a surprising result if we can show that the conformal structure on triple junction spheres is unique.

Moreover, this also has some relations with the minimal triple junction surfaces in sphere. Please see Section 7.6 for details.

7.3 Some examples

To better illustrate uniformization, let us give some examples to explain what would happen after uniformization.

Let us take Σ^1 as the union of a disk and a torus with an open disk removed. Take Σ^2, Σ^3 to be the surfaces diffeomorphic to annular regions. Then we identify their boundaries as shown in the left side of Figure 7.1 to get a triple junction surface Σ. We have used colored curves to mark Γ. We will choose $\theta = (1, 1, 1)$ for Σ and suppose we have fixed a $C^{(1)}$ metric on it.

Remark 7.16. *Although we assume each Σ^i to be connected in definitions of triple junction surfaces, we can allow them to have more than one component. All the previous results are held for this case, including uniformization.*

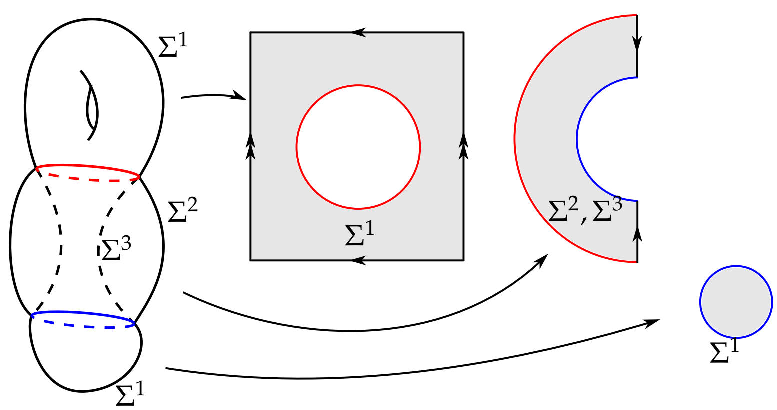


Figure 7.1: *Uniformization of* Σ

Note that $\chi(\Sigma) = 0$. Hence according to Theorem 7.14, we can find a new metric $\overline{g}$, which is conformally equivalent to g such that the Gaussian curvature of $\overline{g}$ is zero. So we may identify each Σ^i as a region in $\mathbb{R}^2$ with standard Euclidean metric. For each Riemann torus, we know it is conformally equivalent to a parallelogram in $\mathbb{R}^2$ by gluing their opposite edges. Hence, after uniformization, we may suppose the torus part of Σ^1 is just a rectangle with a disk removed for simplicity. In Figure 7.1, we have drawn this rectangle, and we use black arrows to mark the way to glue it to form the torus part of Σ^1. For the disk part, we may assume it is a disk in $\mathbb{R}^2$.

For Σ^2, Σ^3, we can suppose they are isometric to each other. Since we have condition $\sum_{i=1}^3 \kappa_{\bar{g}}^i = 0$, we get $\kappa_{\bar{g}}^2 = \kappa_{\bar{g}}^3 = -\frac{1}{2}\kappa_{\bar{g}}^1$. So, each component of $\partial\Sigma^2$ or $\partial\Sigma^3$ will be the half-circle with twice the radius of the corresponding circles in $\partial\Sigma^1$. So we draw a half annular region in $\mathbb{R}^2$ with some boundary part identified as shown in Figure 7.1.

It would be great if every $\bar{g}$ can be described in Figure 7.1 after uniformization. In general, we might not have such a nice result. We do not know whether κ_g will be locally constant on Γ. For a general metric g on Σ, we may only get a weird boundary curve as shown in Figure 7.2

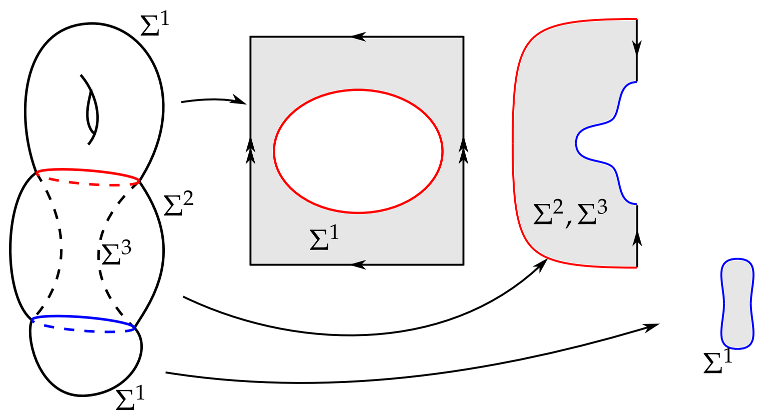


Figure 7.2: *Another possible uniformization result of* Σ

From this example, we may note that the geodesic curvature of Γ in each Σ^i might have many possibilities. Recall that for a closed Riemann surface, we know the class of conformal structure (moduli spaces) on it would have some manifold structure of finite-dimensional. The behavior of the above examples suggests we may not be able to get such finite-dimensional manifold structure anymore. One possible way is to seek a weaker uniformization for triple junction surfaces. This is the topic I would like to talk about in the next section.

7.4 Weak uniformization

Suppose we want to extend the meaning of conformal equivalence, according to the example in the last section. In that case, we might need to change the $C^{(0)}$ triple junction structure on Σ (find another identification on $\mathring{\Gamma}^i$). This thought leads to the following definitions.

Definition 7.17. Let Σ be a triple junction surface with given $C^{(1)}$ metric g. We say a new metric $\overline{g}$ together with three diffeomorphisms, $\overline{\varphi}^i : \Gamma \to \mathring{\Gamma}^i$ is weakly conformal to g if each $\overline{g}^i$ is conformal to g^i on Σ^i and $\overline{g}$ will be a $C^{(1)}$ metric under new $C^{(0)}$ triple junction structure.

Hence, under weak conformal equivalence conditions, we can impose more conditions on the metric $\overline{g}$ after uniformization. For instance, we may want to ensure the geodesic curvature is locally constant on Γ.

This is one of the problems we may ask.

Problem 7.18 (Weak uniformization)**.** *Suppose Σ is a triple junction surface with given $C^{(1)}$ metric g such that $\partial\Sigma = \emptyset$. Can we find a new metric $\overline{g}$ which is weakly conformal to g such that the geodesic curvature of boundary $\kappa^i_{\overline{g}^i}$ is constant on each component of Γ and it has constant Gaussian curvature in the interior of Σ?*

Remark 7.19. *Although we require g to be a $C^{(1)}$ metric on Σ, we may only need to assume it is an arbitrary metric on Σ. This is because the weak uniformization may change the $C^{(0)}$ triple junction structure, so it is not necessary to assume g is $C^{(1)}$. We only need to ensure that the new metric $\overline{g}$ should be a $C^{(1)}$ metric under the new $C^{(0)}$ triple junction structure.*

This problem has been solved if $\chi(\Sigma) \leq 0$ and Γ is connected. This is the main result in [Wan21b]. Let us summarize it here.

Theorem 7.20. *Suppose Σ is a triple junction surface with given $C^{(1)}$ metric and $\partial\Sigma = \emptyset$. We also assume density $\theta = (1,1,1)$ and Γ is connected, then we may find $\overline{g}$ which is weakly conformal to g such that $\overline{g}^i$ is a hyperbolic metric and it has constant geodesic curvature on Γ.*

Remark 7.21. *Although we have assumed $\theta = (1,1,1)$ in Theorem 7.20, we can show that this* *theorem is valid for general density θ on Σ. The proof is an easy modification of the methods in [Wan21b].*

The key steps in the proof of Theorem 7.20. Let $L^i(g)$ denote the length of Γ under metric g^i. In view of the definition of weak uniformization, we only need to find a hyperbolic metric $\overline{g}^i$ on each Σ^i, conformal to g^i, such that $L^i(\overline{g}) = L^j(\overline{g})$ for $1 \leq i, j \leq 3$ and Γ has constant geodesic curvature under metric $\overline{g}^i$ with $\sum_{i=1}^3 \kappa^i_{\overline{g}^i} \equiv 0$. This can be completed by the following theorem.

Theorem 7.22. *Let M be a smooth, compact oriented surface with a connected boundary. We fix a metric g on M. Then for any $L \in (0, +\infty)$, there is a metric $\overline{g}$, conformal to g, such that the following results hold,*

- *The length of boundary ∂M is L.*
- *∂M has constant geodesic curvature $c = c(L) \in \mathbb{R}$.*

Moreover, the function $L \to Lc(L)$ is continuous and strictly increasing with the following limits,

$$\lim_{L\to 0} Lc(L) = 2\pi\chi(M), \quad \lim_{L\to+\infty} Lc(L) = +\infty.$$

The proof for Theorem 7.22 is an extension of the work [Rup21]. Readers may refer to [Wan21b] for details.

Now we define the function $c^i(L)$ for each $L \in (0, +\infty)$ by choosing $M = \Sigma^i$ in Theorem 7.22. Hence, the function $C(L) = \sum_{i=1}^3 Lc^i(L)$ is continuous and strictly increasing with $\lim_{L\to 0} C(L) = 2\pi\chi(\Sigma)$ and $\lim_{L\to\infty} C(L) = +\infty$. Note that $\chi(\Sigma) < 0$, we can find unique $L_0 \in (0, +\infty)$ such that $C(L_0) = 0$.

For such L_0, we can choose $\overline{g}^i$ based on Theorem 7.22 and we know $\sum_{i=1}^3 c^i(L_0) = 0$. Hence $\overline{g} = (\overline{g}^1, \overline{g}^2, \overline{g}^3)$ is the metric we want. $\square$

Remark 7.23. *Note that the weak uniformization for the standard triple junction sphere is trivial. For any g^i on disk, we know it is conformal to the metric that appeared in the standard half sphere.*

Clearly, we can glue the boundary of the standard half sphere to get a triple junction surface. Hence, it has a unique weak conformal structure.

Remark 7.24. *For the weak uniformization, we may conjecture that we can establish a finite-dimensional manifold-like structure on moduli spaces of weak conformal structure on triple junction surfaces.*

7.5 Examples of weak uniformization

In Section 7.3, we have learned an example shown in Figure 7.1.

Here are more examples of the possible results of weak uniformization.

The first one is shown in Figure 7.3. We choose Σ^1, Σ^2 as two isometric torus with a disk removed and choose Σ^3 as a disk. We identify their boundaries to get a triple junction surface Σ.

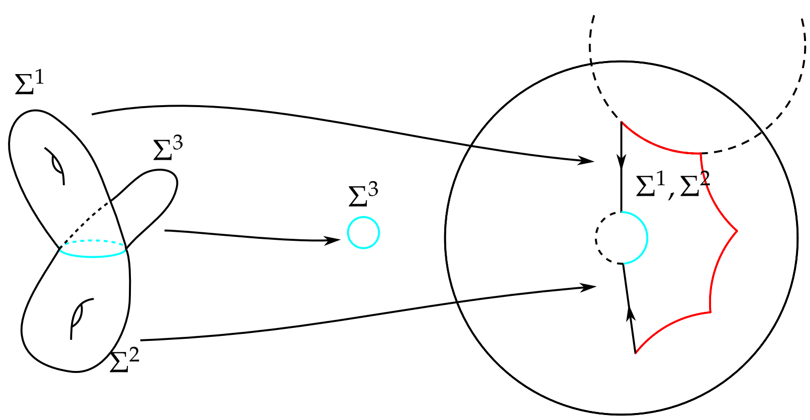


Figure 7.3: *Weak uniformization, first example*

Note that $\chi(\Sigma) = -1$ in this case. So we may identify each Σ^i to be a domain in a hyperbolic space after uniformization. For the surface Σ^1, Σ^2 in a hyperbolic space shown in Figure 7.3, we may glue the black curves to get a parallelogram-type region with a disk removed and then glue its red boundary to get a torus without a disk. We also view Σ^3 as a small disk in hyperbolic space.

The second one is shown in Figure 7.4. We will choose Σ^1 as the genus-two surface with a disk removed and Σ^2 and Σ^3 as the disk. We also have $\chi(\Sigma) = -1$ for this case. These examples show that there are many different types of triple junction surfaces even when they have the same Euler characteristic. We may use same notations to mark the way of uniformization in Figure 7.4.

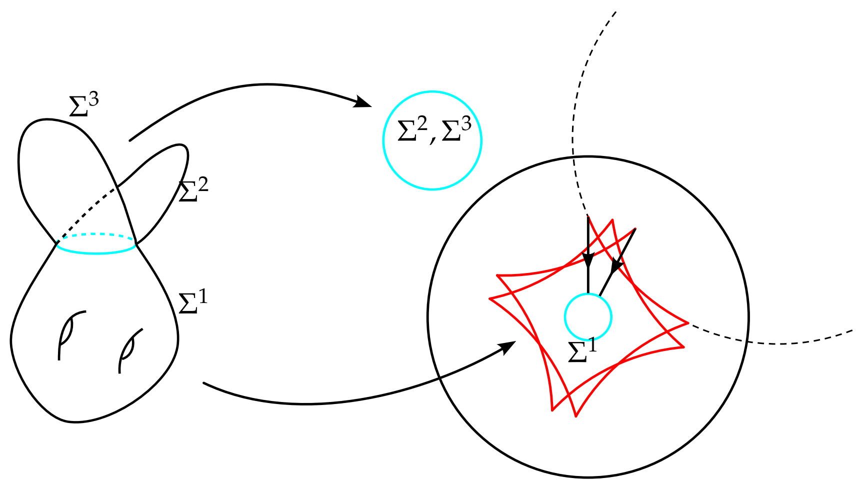


Figure 7.4: *Weak uniformization, second example*

7.6 The first non-trivial eigenvalue of triple junction surfaces

This section shows that we may establish the connection between the first non-trivial eigenvalue and minimal triple junction surfaces in the unit spheres. This part is similar to the result in [LY82]. Since there are many things we do not know regarding the conformal structure, there might be many open problems in this direction.

Let us fix a triple junction surface Σ with a conformal structure J. We will suppose $\partial\Sigma = \emptyset$ in this section. We say a map $\phi : \Sigma \to \mathbb{S}^n$ is conformal if each $\phi^i : \Sigma^i \to \mathbb{S}^n$ is a smooth conformal map and it satisfies $\sum_{i=1}^{3} \theta^i \tau^i = 0$. Here, we will choose τ as the unit outer normal of Γ as its triple junction structure.

Let G be the group of all conformal diffeomorphisms on $\mathbb{S}^n$. We can define the *n-conformal volume* of ϕ as

$$V_J(n,\phi) := \sup_{g\in G} \int_\Sigma \theta \left|\nabla(g\circ\phi)\right|^2 d\Sigma.$$

Geometrically, it is the supremum of the volume of Σ under map $g\circ\phi$ in $\mathbb{S}^n$ with the density θ. We define the *n-conformal volume* of Σ as

$$V_J(n,\Sigma) := \inf_{\phi,\ \text{conformal}} V_J(n,\phi).$$

The first thing we note is that we have a lower bound of the conformal volume as follows.

Proposition 7.25. *For the conformal volume defined above, we have*

$$V_J(n,\Sigma) \geq 2\pi \sum_{i=1}^{3} \theta^i.$$

Proof. To see this point, we can use the conformal map $g_k \in G$ on the sphere to blow up a point on Γ. A similar argument in [LY82] for Fact 2 implies that we can choose a sequence of $g_k \in G$ such that the limit of the image of $g_k\circ\phi$ contains three half-spheres with density θ. So we know $V_J(n,\phi) \geq 2\pi\sum_{i=1}^{3}\theta^i$. This will finish the proof. □

Now, we can state our theorem of the relationship between the first non-trivial eigenvalue and the conformal area.

Theorem 7.26. *Suppose Σ defined above with a $C^{(1)}$ metric g_0 which induces the conformal structure J. We suppose λ is the first non-trivial eigenvalue for the Laplacian defined on space $H^1_1(\Sigma)$ on Σ. Then*

$$\lambda V(\Sigma) \leq 2V_J(n,\Sigma), \tag{7.11}$$

for all n where $V_J(n,\Sigma)$ is well-defined. The equality holds will imply Σ can be minimally immersed into $\mathbb{S}^n$.

Here, we write $V(\Sigma)$ as the volume of Σ under metric g_0 defined as $V(\Sigma) := \int_\Sigma \theta d\Sigma$.

We say an eigenvalue is non-trivial if the corresponding eigenfunction is not a locally constant function. Since the Laplacian has zero as its first eigenvalue, we suppose λ is the second eigenvalue.

Remark 7.27. *The first eigenvalue is simple in this case since we require the operator to be defined on $H_1^1(\Sigma)$.*

Proof of Theorem 7.26. Let ϕ be a conformal map of Σ to $\mathbb{S}^n$ with $V_J(n,\phi) \leq V_J(n,\Sigma) + \varepsilon$ for $\varepsilon > 0$ small.

We write x_k as the coordinate functions on $\mathbb{R}^{n+1}$. Note that by the same balancing argument that appeared in the proof of Theorem 1 in [LY82], we can find an element $g \in G$ such that

$$\int_\Sigma \theta x_k \circ g \circ \phi d\Sigma = 0, \quad \forall 1 \leq k \leq n+1. \tag{7.12}$$

On the other hand, we also have

$$\sum_{k=1}^{n+1} \int_\Sigma \theta (x_k \circ g \circ \phi)^2 d\Sigma = \int_\Sigma \left(\sum_{k=1}^{n+1} (x_k)^2 \right) \circ g \circ \phi \theta d\Sigma = V(\Sigma), \tag{7.13}$$

$$\begin{aligned} \sum_{k=1}^{n+1} \int_\Sigma |\nabla (x_k \circ g \circ \phi)|^2 \theta d\Sigma &= \int_\Sigma \sum_{k=1}^{n+1} (g \circ \phi)^* (\theta |\nabla x_k|^2 d\mathbb{S}^n) \\ = 2 \int_\Sigma (g \circ \phi)^* (\theta d\mathbb{S}^n) &\leq 2V_J(n,\phi) \leq 2(V_J(n,\Sigma) + \varepsilon), \end{aligned} \tag{7.14}$$

where $d\mathbb{S}^n$ denotes the area elements on sphere $\mathbb{S}^n$. Note that we have used each Σ^i is a minimal conformal immersion to get $\sum_{k=1}^n |\nabla x_k|^2 = 2$.

Since the first eigenfunction space only contains constant functions, we know λ can be characterized by

$$\lambda = \inf_{u \in H_1^1(\Sigma), \int_\Sigma u \theta d\Sigma = 0.} \frac{\int_\Sigma |\nabla u|^2 \theta d\Sigma}{\int_\Sigma u^2 \theta d\Sigma}, \tag{7.15}$$

in view of Theorem 5.14.

Hence, by identity (7.12), we can choose u by $x_k \circ g \circ \phi$ and using equations (7.13), (7.14) to get

$$\lambda \sum_{k=1}^{n+1} \int_\Sigma (x_k \circ g \circ \phi)^2 \theta d\Sigma \leq \sum_{k=1}^{n+1} \int_\Sigma |\nabla (x_k \circ g \circ \phi)|^2 \theta d\Sigma,$$

which implies $\lambda V(\Sigma) \leq 2(V_J(n,\Sigma)+\varepsilon)$. Hence,

$$\lambda V(\Sigma) \leq 2V_J(n,\Sigma).$$

Now let us suppose the equality (7.11) holds. Up to a scaling, we suppose $V(\Sigma)=1$. Similar as in the proof that appeared in [LY82] (essentially the same), we may find a sequence of ϕ_j such that the following things hold for some N with $1 \leq N \leq n+1$,

$$\lim_{j\to\infty} V_J(n,\phi_j) = V_J(n,\Sigma), \tag{7.16}$$

$$\int_\Sigma x_k \circ \phi_j \theta d\Sigma = 0, \tag{7.17}$$

$$2V_J(n,\phi_j) \geq \sum_{k=1}^{n+1} \int_\Sigma \left|\nabla x_k \circ \phi_j\right| \theta d\Sigma \geq 2\sum_{k=1}^{n+1} \int_\Sigma (x_k\circ\phi_j)^2 \theta d\Sigma, \tag{7.18}$$

$$\lim_{j\to\infty} \sum_{k=1}^{N} \int_\Sigma (x_k\circ\phi_j)^2\theta d\Sigma = V(\Sigma), \tag{7.19}$$

$$\lim_{j\to\infty} \int_\Sigma (x_k\circ\phi_j)^2\theta d\Sigma = 0 \text{ for } N+1 \leq k \leq n+1. \tag{7.20}$$

In view of (7.16), we may suppose $x_k\circ\phi_j \to \psi_k$ weakly in $H_1^1(\Sigma)$ and strongly in $L^2(\Sigma)$. Hence, the above inequality together with (7.15) implies ψ_k are the eigenfunctions corresponding to the eigenvalue 2 and $\psi_k \equiv 0$ for $N+1 \leq j \leq n+1$. This will imply $\psi_k \in C^{(1)}_{E_1,E_\theta}(\Sigma)$ for each $1\leq k\leq N$ and solve the equation $-\Delta\psi_k = 2\psi_k$.

Note that we have $\sum_{i=1}^N \psi_i^2 = 1$ on Σ, we can choose $\psi := (\psi_1,\cdots,\psi_N)$ to be the smooth conformal immersion of Σ to $\mathbb{S}^{N-1}$. Taking derivative over $|\psi|^2 = 1$ implies $|\nabla\psi|^2 = 2|\psi|^2 = 2$. Hence, ψ will be an isometric map on Σ. Recall that from first variation formula (cf. Subsection 4.4.1), we know an immersion $\phi : \Sigma \to \mathbb{S}^n$ is a minimal triple junction surface if and only if ϕ satisfies the following equations,

$$\begin{cases} -\Delta_\Sigma\phi_k = 2\phi_k, & \text{in } \Sigma, \quad \forall 1\leq k\leq N, \\ \sum_{i=1}^3 \theta^i\tau^i = 0, & \text{on } \Gamma. \end{cases} \tag{7.21}$$

The second condition is equivalent to the condition $\frac{\partial\phi_k}{\partial\tau}|_\Gamma \in \Gamma(E_\theta)$ for each k. Hence, we know ψ is an isometric minimal immersion of Σ. □

In particular, for any triple junction sphere Σ with a conformal structure J, we know it is minimally isometric immersed into a unit sphere $\mathbb{S}^n$ with metric in conformal class J by Theorem 7.26 if it can be conformally immersed into $\mathbb{S}^n$. So the conformal structure on the triple junction sphere is closely related to the minimal triple junction spheres in $\mathbb{S}^n$. If we indeed know the information of conformal structure on Σ, we may answer the uniqueness of minimal triple junction spheres in $\mathbb{S}^n$. Conversely, if we can construct other non-trivial minimal triple junction spheres, we can also get some results about conformal structures on Σ.

References

[AtEKS14] W. Arendt, A.F.M. ter Elst, J.B. Kennedy, and M. Sauter. The Dirichlet-to-Neumann operator via hidden compactness. *Journal of Functional Analysis*, 266(3):1757–1786, February 2014.

[BM21] Jacob Bernstein and Francesco Maggi. Symmetry and rigidity of minimal surfaces with Plateau-like singularities. *Archive for Rational Mechanics and Analysis*, 239(2):1177–1210, 2021.

[Bre02a] Simon Brendle. Curvature flows on surfaces with boundary. *Mathematische Annalen*, 324(3):491–519, 2002.

[Bre02b] Simon Brendle. A family of curvature flows on surfaces with boundary. *Mathematische Zeitschrift*, 241(4):829–869, 2002.

[CdL03] T. H. Colding and C. de Lellis. The min-max construction of minimal surfaces. *Surveys in Differential Geometry*, 8(1):75–107, 2003.

[CHH$^+$94] Cox, Harrison, Hutchings, Kim, Light, Mauer, Tilton, and Morgan. The shortest enclosure of three connected areas in $\mathbb{R}^2$. *Real Analysis Exchange*, 20(1):313–335, 1994.

[CL21] Otis Chodosh and Chao Li. Stable minimal hypersurfaces in $\mathbf{R}^4$. *arXiv preprint arXiv:2108.11462*, August 2021.

[CY88] Sun-Yung A Chang and Paul C Yang. Conformal deformation of metrics on $\mathbb{S}^2$. *Journal of differential geometry*, 27(2):259–296, 1988.

[dCP12] M. do Carmo and C. K. Peng. Stable complete minimal surfaces in $\mathbb{R}^3$ are planes. In *Manfredo P. do Carmo – Selected Papers*, pages 163–166. Springer Berlin Heidelberg, 2012.

[DG13] Daniel Depner and Harald Garcke. Linearized stability analysis of surface diffusion for hypersurfaces with triple lines. *Hokkaido Mathematical Journal*, 42(1):11–52, 2013.

[DGK14] Daniel Depner, Harald Garcke, and Yoshihito Kohsaka. Mean curvature flow with triple junctions in higher space dimensions. *Archive for Rational Mechanics and Analysis*, 211(1):301–334, 2014.

[DLT13] Camillo De Lellis and Dominik Tasnady. The existence of embedded minimal hypersurfaces. *Journal of Differential Geometry*, 95(3):355–388, November 2013.

[Dou31] Jesse Douglas. Solution of the problem of Plateau. *Transactions of the American Mathematical Society*, 33(1):263–321, 1931.

[FC85] Doris Fischer-Colbrie. On complete minimal surfaces with finite Morse index in three manifolds. *Inventiones Mathematicae*, 82(1):121–132, February 1985.

[FCS80] Doris Fischer-Colbrie and Richard Schoen. The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. *Communications on Pure and Applied Mathematics*, 33(2):199–211, March 1980.

[FGB$^+$93] Joel Foisy, Manuel Alfaro Garcia, Jeffrey Brock, Nickelous Hodges, and Jason Zimba. The standard double soap bubble in R^2 uniquely minimizes perimeter. *Pacific journal of mathematics*, 159(1):47–59, 1993.

[Fre10] Alexandre Freire. Mean curvature motion of triple junctions of graphs in two dimensions. *Communications in Partial Differential Equations*, 35(2):302–327, 2010.

[HM05] Aladar Heppes and Frank Morgan*. Planar clusters and perimeter bounds. *Philosophical Magazine*, 85(12):1333–1345, 2005.

[HS21] Han Hong and Artur B. Saturnino. Capillary surfaces: stability, index and curvature estimates. *arXiv preprint arXiv:2105.12662*, 31:11971–11987, May 2021.

[Khu91] Hala Halim Khuri. Heights on the moduli space of Riemann surfaces with circle boundaries. *Duke Mathematical Journal*, 64(3):555–570, 1991.

[Kim08] Young-Heon Kim. Surfaces with boundary: Their uniformizations, determinants of Laplacians, and isospectrality. *Duke Mathematical Journal*, 144(1):73–107, 2008.

[Kru13] Brian Krummel. Regularity of minimal hypersurfaces with a common free boundary. *Calculus of Variations and Partial Differential Equations*, 51(3-4):525–537, November 2013.

[LM96] Gary Lawlor and Frank Morgan. Curvy slicing proves that triple junctions locally minimize area. *Journal of Differential Geometry*, 44(3):514–528, January 1996.

[LR18] Camillo De Lellis and Jusuf Ramic. Min-max theory for minimal hypersurfaces with boundary. *Annales de l'institut Fourier*, 68(5):1909–1986, 2018.

[LY82] Peter Li and Shing-Tung Yau. A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. *Inventiones mathematicae*, 69(2):269–291, 1982.

[LZ21] Martin Man-Chun Li and Xin Zhou. Min-max theory for free boundary minimal hypersurfaces, I: Regularity theory. *Journal of Differential Geometry*, 118(3), July 2021.

[LZZ21] Chao Li, Xin Zhou, and Jonathan J. Zhu. Min-max theory for capillary surfaces. *arXiv preprint arXiv:2111.09924*, November 2021.

[Mal17] Andrea Malchiodi. A variational approach to Liouville equations. *Bollettino dell'Unione Matematica Italiana*, 10(1):75–97, 2017.

[MFG98] Frank Morgan, Christopher French, and Scott Greenleaf. Wulff clusters in $\mathbb{R}^2$. *The Journal of Geometric Analysis*, 8(1):97–115, 1998.

[MP04] William Hamilton Meeks and Joaquín Pérez. Conformal properties in classical minimal surface theory. *Surveys in Differential Geometry*, 9(1):275–335, 2004.

[MT02] Rafe Mazzeo and Michael Taylor. Curvature and uniformization. *Israel Journal of Mathematics*, 130(1):323–346, 2002.

[MW02] Frank Morgan and Wacharin Wichiramala. The standard double bubble is the unique stable double bubble in $\mathbf{R}^2$. *Proceedings of the American Mathematical Society*, 130(9):2745–2751, 2002.

[MY06] Chikako Mese and Sumio Yamada. The parameterized Steiner problem and the singular Plateau problem via energy. *Transactions of the American Mathematical Society*, 358(7):2875–2895, 2006.

[Nic20] Liviu I Nicolaescu. *Lectures on the Geometry of Manifolds*. World Scientific, 2020.

[OPS88] Brad Osgood, Ralph Phillips, and Peter Sarnak. Extremals of determinants of Laplacians. *Journal of Functional Analysis*, 80(1):148–211, 1988.

[OPS89] Brad Osgood, Ralph Phillips, and Peter Sarnak. Moduli space, heights and isospectral sets of plane domains. *Annals of Mathematics*, 129:293, 1989.

[Oss86] Robert Osserman. *A survey of minimal surfaces*. Dover Publications, New York, 1986.

[Pog81] Aleksei Vasil'evich Pogorelov. On the stability of minimal surfaces. In *Doklady Akademii Nauk*, volume 260, pages 293–295. Russian Academy of Sciences, 1981.

[Ros93] Harold Rosenberg. Hypersurfaces of constant curvature in space forms. *Bull. Sci. Math*, 117(2):211–239, 1993.

[RS97] Antonio Ros and Rabah Souam. On stability of capillary surfaces in a ball. *Pacific Journal of Mathematics*, 178(2):345–361, April 1997.

[Rup21] Melanie Rupflin. Hyperbolic metrics on surfaces with boundary. *The Journal of Geometric Analysis*, 31(3):3117–3136, 2021.

[Sch83] Richard M. Schoen. Uniqueness, symmetry, and embeddedness of minimal surfaces. *Journal of Differential Geometry*, 18(4):791–809, 1983.

[Sim93] Leon Simon. Cylindrical tangent cones and the singular set of minimal submanifolds. *Journal of Differential Geometry*, 38(3):585–652, 1993.

[Smi83] Francis R. Smith. On the existence of embedded minimal 2-spheres in the 3-sphere, endowed with an arbitrary metric. *Bulletin of the Australian Mathematical Society*, 28(1):159–160, August 1983.

[SS81] Richard Schoen and Leon Simon. Regularity of stable minimal hypersurfaces. *Communications on Pure and Applied Mathematics*, 34(6):741–797, November 1981.

[SSY75] Richard Schoen, Leon Simon, and Shing-Tung Yau. Curvature estimates for minimal hypersurfaces. *Acta Mathematica*, 134(0):275–288, 1975.

[Str05] Michael Struwe. A flow approach to Nirenberg's problem. *Duke Mathematical Journal*, 128(1):19–64, 2005.

[SW20] Felix Schulze and Brian White. A local regularity theorem for mean curvature flow with triple edges. *Journal für die reine und angewandte Mathematik*, 2020(758):281–305, 2020.

[Tay73] Jean E. Taylor. Regularity of the singular sets of two-dimensional area-minimizing flat chains modulo 3 in R^3. *Inventiones Mathematicae*, 22(2):119–159, June 1973.

[Tay76] Jean E. Taylor. The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. *Annals of Mathematics*, 103(3):489, 1976.

[Tay11] Michael E. Taylor. *Partial Differential Equations III*. Springer New York, 2011.

[Tra20] Hung Tran. Index characterization for free boundary minimal surfaces. *Communications in Analysis and Geometry*, 28(1):189–222, 2020.

[Wan21a] Gaoming Wang. Index of embedded networks in the sphere. *arXiv preprint arXiv:2110.12653*, October 2021.

[Wan21b] Gaoming Wang. Uniformization of surfaces with boundary and the application to the triple junction surfaces with negative euler characteristic. *arXiv preprint arXiv:2110.12656*, October 2021.

[Wan22] Gaoming Wang. Curvature estimates for stable minimal surfaces with a common free boundary. *Calculus of Variations and Partial Differential Equations*, 61(4), May 2022.

[Wic04] Wacharin Wichiramala. Proof of the planar triple bubble conjecture. *Journal fur die Reine und Angewandte Mathematik*, 2004:1–49, 2004.

[Yau82] Shing-Tung Yau. *SURVEY ON PARTIAL DIFFERENTIAL EQUATIONS IN 3 DIFFERENTIAL GEOMETRY*, volume 102. Princeton University Press Princeton, NJ, 1982.